



Lie Algebras/Harmonic Analysis

Deformation quantization and invariant differential operators

Quantification par déformation et opérateurs différentiels invariants

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ABSTRACT

In this Note we explain how the techniques of deformation quantization in the sense of Kontsevich can be used to describe the algebra of invariant differential operators on Lie groups.

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R É S U M É

Nous expliquons dans cette Note comment les méthodes de quantification par déformation, au sens de Kontsevich, peuvent être utilisées pour décrire l'algèbre des opérateurs différentiels invariants sur un groupe de Lie.

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Version française abrégée

Soit \mathfrak{g} une algèbre de Lie, $\mathfrak{h} \subset \mathfrak{g}$ une sous-algèbre et λ un caractère de \mathfrak{h} . Motivé par la quantification par déformation de Kontsevich et la généralisation de Cattaneo–Felder, on considère $\mathfrak{h}_\lambda^\perp$ comme sous-variété coisotrope de la variété de Poisson \mathfrak{g}^* . C'est l'approche de Cattaneo–Torossian des algèbres de Lie pour calculer dans l'algèbre déformée $(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_\lambda)^\mathfrak{h}$. Notre étude est également motivée par les Conjectures de Duflo et Corwin–Greenleaf. Notre point de départ est l'isomorphisme $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^\mathfrak{h} \simeq \mathbb{D}(\mathfrak{g}, \mathfrak{h}, \lambda)$ de Koornwinder qui interprète l'algèbre $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^\mathfrak{h}$ comme celle des opérateurs laissant invariant l'espace $C^\infty(G, H, \lambda)$ des fonctions complexes $\theta : G \rightarrow \mathbb{C}$ satisfaisant $\theta(g \cdot \exp X) = e^{-i\lambda(X)}\theta(g)$, $\forall X \in \mathfrak{h}$, $\forall g \in G$.

Nous montrons qu'il existe un isomorphisme non canonique

$$\bar{\beta}_{\mathfrak{q}, (\epsilon)} \circ \partial_{\frac{1}{2} q(\epsilon)} \circ \bar{T}_1^{-1} T_2 : H_{(\epsilon)}^0(\mathfrak{h}_\lambda^\perp, d_{\mathfrak{h}_\lambda^\perp, \mathfrak{q}}^{(\epsilon)}) \xrightarrow{\cong} (U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_\lambda)^\mathfrak{h}$$

entre l'algèbre de réduction sur l'espace affine $\mathfrak{h}_\lambda^\perp := \{f \in \mathfrak{g}^* / f|_{\mathfrak{h}} = -\lambda\}$ et l'algèbre déformée $(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_\lambda)^\mathfrak{h}$. On étudie ensuite la spécialisation $H_{(\epsilon=1)}^0(\mathfrak{h}_\lambda^\perp, d_{\mathfrak{h}_\lambda^\perp, \mathfrak{q}}^{(\epsilon=1)})$, l'algèbre de réduction $H^0(\mathfrak{h}_\lambda^\perp, d_{\mathfrak{h}_\lambda^\perp, \mathfrak{q}})$ définie sans le paramètre de déformation ϵ , et $H^0(\mathfrak{h}_{t\lambda}^\perp, d_{\mathfrak{h}_{t\lambda}^\perp, \mathfrak{q}})$, $t \in \mathbb{R}$ l'algèbre de réduction sans ϵ et en déformant le caractère λ .

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Nous comparons les objets en correspondance entre $H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda, d_{\mathfrak{h}^\perp_\lambda, \mathfrak{q}}^{(\epsilon)})$ et $(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_\lambda)^\mathfrak{h}$ c'est-à-dire $\mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda)$ et $((U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_{t\lambda})^\mathfrak{h})$. On aboutit finalement à

$$\mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda) \stackrel{alg}{\simeq} H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_\lambda, d_{\mathfrak{h}^\perp_\lambda, \mathfrak{q}}^{(\epsilon=1)}).$$

1. Introduction

Let G be a nilpotent, connected and simply connected Lie group, $H \subset G$ a subgroup and $\lambda \in \mathfrak{g}^*$ a character of \mathfrak{h} . Let $C^\infty(G, H, \lambda)$ be the vector space of C^∞ functions $\theta : G \rightarrow \mathbb{C}$ that satisfy the condition

$$\theta(g \cdot \exp X) = e^{-i\lambda(X)}\theta(g), \quad \forall X \in \mathfrak{h}, \forall g \in G$$

and $\mathbb{D}(\mathfrak{g}, \mathfrak{h}, \lambda)$ the algebra of linear differential operators that leave the space $C^\infty(G, H, \lambda)$ invariant and commute with the left translation on G ,

$$D(C^\infty(G, H, \lambda)) \subset C^\infty(G, H, \lambda), \quad \text{and} \quad D(L(g)\theta) = L(g)(D(\theta)).$$

Let also $m(\tau)$ be the multiplicities in the spectral decomposition of the representation τ_λ

$$\tau_\lambda \cong \int_{\hat{G}} m(\tau) \tau \, d\mu(\tau) \cong \int_{(f+\mathfrak{h}^\perp)/H} \tau_l \, d\nu(l)$$

where τ_λ is the representation $Ind(G, H, \lambda) = L^2(G, H, \lambda)$ and \hat{G} is the space of irreducible representations, or otherwise the unitary dual of G . Furthermore ν is a finite and positive measure equivalent to the Lebesgue measure on the affine space $\lambda + \mathfrak{h}^\perp$. Finally $\mu = K_*(\nu)$ (where $K : \mathfrak{g}^* \rightarrow \hat{G}$ is the Kirillov map) is our measure on \hat{G} .

The Corwin–Greenleaf conjecture says that

$$\mathbb{D}(\mathfrak{g}, \mathfrak{h}, \lambda) \text{ is commutative} \iff m(\tau) < +\infty \quad \mu\text{-a.e.}$$

To state a second conjecture of our interest, let \mathfrak{g} be a nilpotent Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra and λ a character of \mathfrak{h} . Let $U(\mathfrak{g})\mathfrak{h}_\lambda$ be the left ideal of $U(\mathfrak{g})$ generated by the family $\langle X + \lambda(X), X \in \mathfrak{h} \rangle$.

Then the Duflo conjecture says that

$$\text{The algebras } C_{poiss}((S(\mathfrak{g})/S(\mathfrak{g})\mathfrak{h}_\lambda)^\mathfrak{h}) \text{ and } C_{ass}[(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^\mathfrak{h}] \text{ are isomorphic.}$$

The first center refers to the Poisson algebra structure of $(S(\mathfrak{g})/S(\mathfrak{g})\mathfrak{h}_\lambda)^\mathfrak{h}$ while the second to the associative structure of $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^\mathfrak{h}$.

The analytic interest of the second conjecture stems from a theorem of Koornwider stating that $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^\mathfrak{h} \simeq \mathbb{D}(\mathfrak{g}, \mathfrak{h}, \lambda)$. Taking into consideration the interpretation of $U(\mathfrak{g})$ as the linear differential operators on G , the Duflo conjecture is actually a question about the structure of the invariant differential operators on the homogeneous space G/H .

In this note we will sketch the proof of the fact that $(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_\lambda)^\mathfrak{h}$ is isomorphic to the reduction algebra related to these data, an algebra playing a central role to the deformation quantization theory. We will also provide results of the same nature for related reduction algebras. The results of this note were presented in [1].

2. Deformation quantization and generalizations

We start by recalling the standard result:

Theorem 1. (See [6].) *Let π be a Poisson bivector of \mathbb{R}^k and $F, G \in C^\infty(\mathbb{R}^k)$. The product*

$$F *_K G := F \cdot G + \sum_{n=1}^{\infty} \epsilon^n \left(\frac{1}{n!} \sum_{\Gamma \in \mathbf{Q}_{n, \bar{2}}} \omega_\Gamma B_{\Gamma, \pi}(F, G) \right)$$

is associative.

The set $\mathbf{Q}_{n, \bar{2}}$ is a special family of graphs Γ . To every admissible graph Γ corresponds a bidifferential operator $B_\Gamma(F, G) := \sum_{R, S} b_{\Gamma}^{R, S} \partial_R(F) \partial_S(G)$ on $C^\infty(\mathbb{R}^k) \times C^\infty(\mathbb{R}^k)$. Note that the functions $b_{\Gamma}^{R, S}$ depend on Γ and are n -linear in the bivector π . The coefficient $\omega_\Gamma \in \mathbb{R}$ is computed by integrating a differential form Ω_Γ (which is also encoded in Γ) over a concentration manifold

$$\hat{C}_{n, \bar{m}}^+ = \{ (z_1, \dots, z_n, z_{\bar{1}}, \dots, z_{\bar{m}}) / z_i \in \mathbb{C}, \Im(z_i) > 0, z_i \in \mathbb{R}, z_i < z_j \text{ for } i < j \} / G_2,$$

where G_2 is the 2-dimensional Lie group of dilations $\langle z_k \mapsto az_k + b, a > 0, b \in \mathbb{R} \rangle$. So $\hat{C}_{n, \bar{m}}^+ \subset (H^+)^n \times \mathbb{R}^m$.

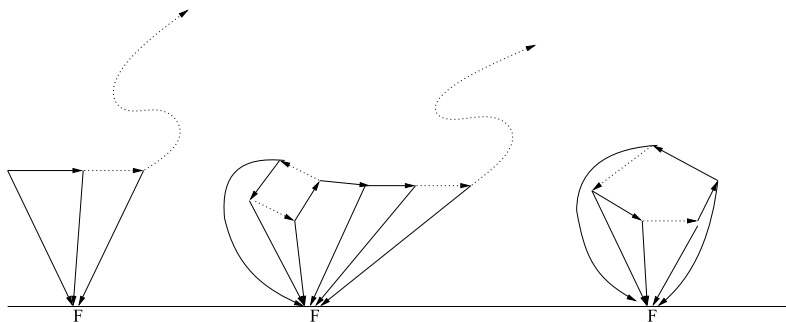


Fig. 1. A Bernoulli graph (B_3) in $d_{\mathfrak{h}^\perp, q}^2$, a Bernoulli-attached-to-a-wheel (BW_7) in $d_{\mathfrak{h}^\perp, q}^2$, and a wheel-type graph.

In the general setting that we briefly recalled, the central result of [6] is the following:

Theorem 2. (See [6].) Let $\mathcal{U} : \mathcal{T}_{poly}(\mathbb{R}^k) \rightarrow \mathcal{D}_{poly}(\mathbb{R}^k)$ be the map defined by the Taylor coefficients:

$$\mathcal{U}_n := \sum_{\bar{m} \geq 0} \left(\sum_{\Gamma \in \mathbf{Q}_{n, \bar{m}}} \omega_\Gamma B_\Gamma \right).$$

Then \mathcal{U} is an L_∞ -morphism and a quasi-isomorphism between the differential graded Lie algebras (DGLA) $\mathcal{T}_{poly}(\mathbb{R}^k)$ of polyvector fields on \mathbb{R}^k and $\mathcal{D}_{poly}(\mathbb{R}^k)$ of polydifferential operators on \mathbb{R}^k .

The key point in the proof is a Stokes equation integrating the form Ω_Γ on $\hat{C}_{n, \bar{m}}^+$. In this way Kontsevich also reached the Duflo isomorphism (applying this to the case of the Poisson manifold \mathfrak{g}^*). Our approach is using the following generalization of what is said, so far. Let X be a Poisson manifold, and $C \subset X$ a coisotropic submanifold. Let NC be the normal bundle, $\mathcal{A} = \bigoplus_{i=0}^{rank(\pi)} \mathcal{A}^i$ with $\mathcal{A}^i = \Gamma(C, \wedge^i NC)$ be the graded commutative algebra of sections of the exterior algebra of the normal bundle NC and $\tilde{\mathcal{D}}(\mathcal{A}) = \bigoplus_n \tilde{\mathcal{D}}^n(\mathcal{A})$ with $\tilde{\mathcal{D}}^n(\mathcal{A}) = \prod_{p+q-1=n} Hom^p(\otimes^q \mathcal{A}, \mathcal{A})$. Finally we set

$$\mathcal{B} = \bigoplus_{j=0}^{\infty} \mathcal{B}^j, \quad \mathcal{B}^0 = \Gamma(C, S((NC)^*)), \quad \mathcal{B}^j = 0, \quad \text{if } j \neq 0.$$

Theorem 3. (See [3].) Consider the DGLA $\mathcal{T}(X, C)$ and $\tilde{\mathcal{D}}(\mathcal{A})$ of polyvector fields and polydifferential operators on X . There is an L_∞ -quasi-isomorphism

$$\mathcal{F}_R : \mathcal{T}(X, C) \rightarrow \tilde{\mathcal{D}}(\mathcal{A}),$$

whose first Taylor coefficient \mathcal{F}_R^1 is the composition

$$F_{HKR} \circ \hat{F} : \mathcal{T}(X, C) \simeq \tilde{\mathcal{T}}(\mathcal{B}) \xrightarrow{\hat{F}} \tilde{\mathcal{T}}(\mathcal{A}) \xrightarrow{F_{HKR}} \tilde{\mathcal{D}}(\mathcal{A})$$

where F_{HKR} is the Hochschild–Konstant–Rosenberg map

$$F_{HKR} := S_{\mathcal{A}}(Der(\mathcal{A})[-1]) \rightarrow \bigoplus_{j=0}^{\infty} Hom_{\mathbf{K}}(\otimes^j \mathcal{A}, \mathcal{A})$$

and \hat{F} is a kind of Fourier transform (for more details we refer to [3]).

We will now apply the previous results for \mathfrak{g}^* with the natural Poisson structure coming from the Lie structure and the orthogonal space \mathfrak{h}^\perp of a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ as a coisotropic submanifold. Some modifications in the theory are needed since the conjectures of the introduction refer to the affine space $-\lambda + \mathfrak{h}^\perp \subset \mathfrak{g}^*$.

3. Applications for non-commutative harmonic analysis

3.1. Reduction equations

Let $\mathfrak{h}^\perp := \{l \in \mathfrak{g}^* / l(\mathfrak{h}) = 0\}$, $\mathfrak{h}_\lambda^\perp := \{f \in \mathfrak{g}^* / f|_{\mathfrak{h}} = -\lambda\}$ and \mathfrak{q} a supplementary space of \mathfrak{h} in \mathfrak{g} . Consider the differential $d_{\mathfrak{h}^\perp, \mathfrak{q}}^{(\epsilon)} : S(\mathfrak{q})[\epsilon] \rightarrow S(\mathfrak{q})[\epsilon] \otimes \mathfrak{h}^*$ where $d_{\mathfrak{h}^\perp, \mathfrak{q}}^{(\epsilon)} := \sum_{i=1}^{\infty} \epsilon^i q_{\mathfrak{h}^\perp, \mathfrak{q}}^{(i)}$ and $d_{\mathfrak{h}^\perp, \mathfrak{q}}^{(i)} = \sum_{\Gamma \in \mathcal{B}_i \cup \mathcal{BW}_i} \bar{\omega}_\Gamma B_\Gamma$.

The second sum is on two families of graphs in $Q_{1,\bar{i}}$: the set \mathcal{B}_i of Bernoulli graphs and the set $\mathcal{B}\mathcal{W}_i$ of Bernoulli-attached-to-a-wheel graph (see Fig. 1). The first terms of the differential $d_{\mathfrak{h}^\perp, \mathfrak{q}}(F) = 0$ are

$$d_{\mathfrak{h}^\perp, \mathfrak{q}}^1(F_n) = 0, \quad d_{\mathfrak{h}^\perp, \mathfrak{q}}^3(F_n) + d_{\mathfrak{h}^\perp, \mathfrak{q}}^1(F_{n-2}) = 0, \quad d_{\mathfrak{h}^\perp, \mathfrak{q}}^5(F_n) + d_{\mathfrak{h}^\perp, \mathfrak{q}}^3(F_{n-2}) + d_{\mathfrak{h}^\perp, \mathfrak{q}}^1(F_{n-4}) = 0, \quad \dots$$

For example, $d_{\mathfrak{h}^\perp, \mathfrak{q}}^1(F_n) = 0 \Rightarrow F_n \in S(\mathfrak{q})^{\mathfrak{h}}$.

Definition 4 (Reduction algebra). (See [1, § 2.3].) We define the reduction space $H_{(\epsilon)}^0(\mathfrak{h}^\perp, d_{\mathfrak{h}^\perp, \mathfrak{q}}^{(\epsilon)})$ of polynomials in the formal variable ϵ as the vector space of polynomials – solutions of the system of linear differential equations

$$d_{\mathfrak{h}^\perp, \mathfrak{q}}^{(\epsilon)}(F(\epsilon)) = 0, \quad F(\epsilon) \in S(\mathfrak{q})[\epsilon].$$

The reduction algebra is $H_{(\epsilon)}^0(\mathfrak{h}^\perp, d_{\mathfrak{h}^\perp, \mathfrak{q}}^{(\epsilon)})$ equipped with the product $*_{CF, \epsilon}$.

We now explain our main results. Let G be a real Lie group, $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra, $\lambda \in \mathfrak{h}^*$ a character of \mathfrak{h} and $f \in \mathfrak{g}^*$ s.t. $f|_{\mathfrak{h}} = \lambda$. Let also \mathfrak{q} be a supplementary space of \mathfrak{h} in \mathfrak{g} .

We set the deformed tensor algebra to be $T_{(\epsilon)}(\mathfrak{g}) := \mathbf{R}[\epsilon] \otimes T(\mathfrak{g})$. Let \mathcal{I}_ϵ be the two-sided ideal $\langle X \otimes Y - Y \otimes X - \epsilon[X, Y] \rangle$ of $T_{(\epsilon)}(\mathfrak{g})$. Since our basic model of $U(\mathfrak{g})$ is $T(\mathfrak{g})$ factored by the non-homogeneous ideal $\langle X \otimes Y - Y \otimes X - [X, Y] \rangle$, we define the deformed universal enveloping algebra of \mathfrak{g} as $U_{(\epsilon)}(\mathfrak{g}) := T_{(\epsilon)}(\mathfrak{g})/\mathcal{I}_\epsilon$. Let also $\mathfrak{h}_\lambda := \{H + \lambda(h), H \in \mathfrak{h}\}$.

Theorem 5. (See [1, § 3.4.3, Theorem 3.1].) The map

$$\bar{\beta}_{\mathfrak{q}, (\epsilon)} \circ \partial_{\frac{1}{q(\epsilon)}} \circ \bar{T}_1^{-1} T_2 : H_{(\epsilon)}^0(\mathfrak{h}_\lambda^\perp, d_{\mathfrak{h}_\lambda^\perp, \mathfrak{q}}^{(\epsilon)}) \xrightarrow{\cong} (U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$$

is an algebra isomorphism.

Here $\bar{\beta}_{\mathfrak{q}, (\epsilon)} : S(\mathfrak{q})[\epsilon] \rightarrow U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_\lambda$ is the symmetrization map, $q(Y) = \det_{\mathfrak{g}}(\frac{\sinh \frac{ad_Y}{2}}{2})$ and T_1, T_2 are differential operators that can be described in terms of Kontsevich graphs [1, § 2.5.2]. The theorem is powerful since no condition is needed for the original Lie group G .

Proof. We use the $H_{(\epsilon)}^0(\mathfrak{h}_\lambda^\perp, d_{\mathfrak{g}^*, \mathfrak{h}_\lambda^\perp, \mathfrak{q}}^{(\epsilon)})$ -bimodule structure

$$T_1 : H_{(\epsilon)}^0(\mathfrak{g}^*, d_{\mathfrak{g}^*}^{(\epsilon)}) \rightarrow H_{(\epsilon)}^0(\mathfrak{h}_\lambda^\perp, d_{\mathfrak{g}^*, \mathfrak{h}_\lambda^\perp, \mathfrak{q}}^{(\epsilon)}), \quad G \mapsto G * 1$$

and

$$T_2 : H_{(\epsilon)}^0(\mathfrak{h}_\lambda^\perp, d_{\mathfrak{h}_\lambda^\perp, \mathfrak{q}}^{(\epsilon)}) \rightarrow H_{(\epsilon)}^0(\mathfrak{h}_\lambda^\perp, d_{\mathfrak{g}^*, \mathfrak{h}_\lambda^\perp, \mathfrak{q}}^{(\epsilon)}), \quad F \mapsto 1 * F$$

in the biquantization diagram (see [4] and [5]) of \mathfrak{g}^* and $\mathfrak{h}_\lambda^\perp$. We show first that $H_{(\epsilon)}^0(\mathfrak{h}_\lambda^\perp, d_{\mathfrak{h}_\lambda^\perp, \mathfrak{q}}^{(\epsilon)}) \subset (U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$ exploiting results from [2,3,5] and the bimodule structure we mentioned to pass from $H_{(\epsilon)}^0(\mathfrak{h}_\lambda^\perp, d_{\mathfrak{h}_\lambda^\perp, \mathfrak{q}}^{(\epsilon)})$ to $(U_{(\epsilon)}(\mathfrak{g})/U_{(\epsilon)}(\mathfrak{g})\mathfrak{h}_\lambda)^{\mathfrak{h}}$.

For the inverse let $H \in \mathfrak{h}$ be a function of first degree $F \in H_{(\epsilon)}^0(\mathfrak{h}_\lambda^\perp, d_{\mathfrak{h}_\lambda^\perp, \mathfrak{q}}^{(\epsilon)})$. We check the graphs in the expression

$$(H + \lambda(H)) * 1 * F.$$

We show that $(H + \lambda(H)) * 1 (1 * F) = [(H + \lambda(H)) * 1] * F = 0$.

Then examining the possible and admissible graphs in the concentration manifolds in this expression we end up to the equation

$$\sum_{\Gamma_{\text{int}}^\alpha, \Gamma_{\text{ext}}^\alpha} \int_0^\infty \hat{\omega}_{\Gamma_{\text{ext}}^\alpha}(s) \mathcal{B}_{\Gamma_{\text{ext}}^\alpha}(\omega_{\Gamma_{\text{int}}^\alpha} \mathcal{B}_{\Gamma_{\text{int}}^\alpha}) ds = 0$$

which is equivalent to

$$\sum_{\alpha} \left(\sum_{\Gamma_{\text{int}}^\alpha, \Gamma_{\text{ext}}^\alpha} \left(\sum_{l, k, m} (\mathcal{B}_{\Gamma_{\text{ext}}^\alpha}^m (\mathcal{B}_{\Gamma_{\text{int}}^\alpha}^k (F_l))) \epsilon^{m+k+l} \right) \right) = 0.$$

This last equation gives (after determining the admissible graphs above) the reduction equations defining $H_{(\epsilon)}^0(\mathfrak{h}_\lambda^\perp, d_{\mathfrak{h}_\lambda^\perp, \mathfrak{q}}^{(\epsilon)})$. \square

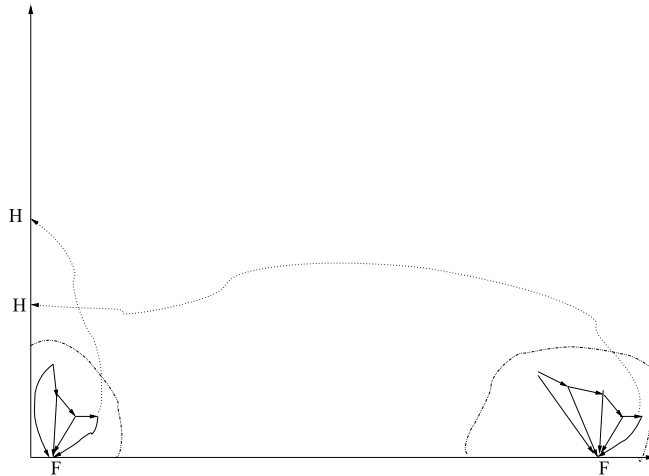


Fig. 2. Behavior of the expression $(H + \lambda(H)) * 1 * 2 F$ when $s \rightarrow 0$ and $s \rightarrow \infty$.

3.2. The specialization algebra $H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_\lambda, d_{\mathfrak{h}^\perp_\lambda, q}^{(\epsilon=1)})$

Definition 6. The specialization algebra for the affine space $-\lambda + \mathfrak{h}^\perp = \mathfrak{h}^\perp_\lambda$, is defined as

$$H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_\lambda, d_{\mathfrak{h}^\perp_\lambda, q}^{(\epsilon=1)}) := (H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda, d_{\mathfrak{h}^\perp_\lambda, q}^{(\epsilon)})) / (\epsilon - 1).$$

The Cattaneo–Felder product on $H^0_{(\epsilon=1)}(\mathfrak{h}^\perp_\lambda, d_{\mathfrak{h}^\perp_\lambda, q}^{(\epsilon=1)})$ will be also denoted as $*_{CF, (\epsilon=1)}$.

We have to note here that one may consider also the reduction algebras $H^0(\mathfrak{h}^\perp_{t\lambda}, d_{\mathfrak{h}^\perp_{t\lambda}, q})$ (that is defined without the formal variable ϵ) and $H^0(\mathfrak{h}^\perp_{t\lambda}, d_{\mathfrak{h}^\perp_{t\lambda}, q}), t \in \mathbb{R}$ (deforming the character λ).

Let now $F' \in H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda, d_{\mathfrak{h}^\perp_\lambda, q}^{(\epsilon)})$. Let $J : H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda, d_{\mathfrak{h}^\perp_\lambda, q}^{(\epsilon)}) \rightarrow H^0(\mathfrak{h}^\perp_\lambda, d_{\mathfrak{h}^\perp_\lambda, q}), J(F') := \sum_k F'_k$. To describe the image of the map J we have the following:

Theorem 7. (See [1, § 3.5.2].) • Let $F \in H^0(\mathfrak{h}^\perp_\lambda, d_{\mathfrak{h}^\perp_\lambda, q})$. Suppose an element $F_{(t)} = \sum_p t^p F_p$ with $F_{(t)} \in H^0(\mathfrak{h}^\perp_{t\lambda}, d_{\mathfrak{h}^\perp_{t\lambda}, q}), \forall t \in \mathbb{R}^*$, and $F_{(t=1)} = F$. Let $F_p = \sum_i F_p^{(i)}$ be a decomposition to homogeneous components and $F_{(\epsilon)} := \epsilon^N \sum F_p^{(i)} \frac{1}{\epsilon^{i+p}} (N \gg \max(i + p))$. Then $F_{(\epsilon)} \in H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda, d_{\mathfrak{h}^\perp_\lambda, q}^{(\epsilon)})$ and $J(F_{(\epsilon)}) = F$.

• Let $F \in H^0(\mathfrak{h}^\perp_\lambda, d_{\mathfrak{h}^\perp_\lambda, q})$. Suppose an element $F_{(\epsilon)} = \sum_{0 \leq k \leq n} \epsilon^k F_k$ with $F_{(\epsilon)} \in H^0_{(\epsilon)}(\mathfrak{h}^\perp_\lambda, d_{\mathfrak{h}^\perp_\lambda, q}^{(\epsilon)})$ and $J(F_{(\epsilon)}) = F$. Let $F_k = \sum_i F_k^{(i)}$ be a decomposition to homogeneous components and $F_{(t)} := t^N \sum_{i,k} \frac{1}{t^{i+k}} F_k^{(i)} (N \gg \max(i + k))$. Then $\forall t \in \mathbb{R}^*, F_{(t)} \in H^0(\mathfrak{h}^\perp_{t\lambda}, d_{\mathfrak{h}^\perp_{t\lambda}, q})$.

3.3. Deformations

In this section we will make clear the relation between the various reduction algebras presented in the previous part. Using Theorem 5 we will associate them to the appropriate algebras of operators. Let $\mathfrak{g}_T := \mathfrak{g} \oplus \mathbb{R}T, [\mathfrak{g}, T] = 0$. We note as $\mathcal{P}_{(t)}((U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_{t\lambda})^\mathfrak{h})$ the algebra of polynomial families in $t, t \rightarrow u_t \in (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_{t\lambda})^\mathfrak{h}$. Let $e_t : U(\mathfrak{g}_T) \rightarrow U(\mathfrak{g})$ be the surjective map $T \mapsto t$ and $\forall X \in \mathfrak{g}, X \mapsto X$. Then $(U(\mathfrak{g}_T)/U(\mathfrak{g}_T)\mathfrak{h}_\lambda^T) \simeq U(\mathfrak{g})$ and evaluating at $T = t$, we take the surjective map

$$e_{(T=t)} : ((U(\mathfrak{g}_T)/U(\mathfrak{g}_T)\mathfrak{h}_\lambda^T)^{\mathfrak{h}T} / (T - t)) \hookrightarrow (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_{t\lambda})^\mathfrak{h}.$$

It turns out that if $t \rightarrow u_t \in (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_{t\lambda})^\mathfrak{h}$ is a polynomial family in t , then there is a $u_T \in (U(\mathfrak{g}_T)/U(\mathfrak{g}_T)\mathfrak{h}_\lambda^T)^{\mathfrak{h}T}$ s.t. $e_t(u_T) = u_t$.

Now let's examine the notion of specialization from the differential operator point of view. Let $\mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda) := (U(\mathfrak{g}_T)/U(\mathfrak{g}_T)\mathfrak{h}_\lambda^T)^{\mathfrak{h}T} / (T - 1)$. In the case $t = 1$ of the previous example we get

$$\mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda) \hookrightarrow (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^\mathfrak{h}.$$

More specifically if $u \in \mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda)$ then there is a $u_T \in U(\mathfrak{g}_T)$ s.t. $u = \pi_{(T=1)}(u_T)$. The element $u_t := e_{(T=t)}(u_T) \in (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_{t\lambda})^\mathfrak{h}$ defines a polynomial family in t so $u_t \in \mathcal{P}_{(t)}((U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_{t\lambda})^\mathfrak{h})$. Then

$$\mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda) \simeq \mathcal{P}_{(t=1)}((U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\lambda)^\mathfrak{h}).$$

We get similar results for specialization from the reduction algebra point of view.

Theorem 8. (See [1, § 3.5.3].) • Let $t \mapsto F_t \in H^0(\mathfrak{h}_{t\lambda}^\perp, d_{\mathfrak{h}_{t\lambda}^\perp, \mathfrak{q}})$ be a polynomial family in t . Then there is an $F_T \in H^0((\mathfrak{h}_\lambda^T)^\perp, d_{(\mathfrak{h}_\lambda^T)^\perp, \mathfrak{q}})$ s.t. $e_t(F_T) = F_t$.

$$\bullet \mathbb{D}_{(T=1)}(\mathfrak{g}, \mathfrak{h}, \lambda) \stackrel{\text{alg}}{\simeq} H_{(\epsilon=1)}^0(\mathfrak{h}_\lambda^\perp, d_{\mathfrak{h}_\lambda^\perp, \mathfrak{q}}^{(\epsilon=1)}).$$

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