Logic

## On a theorem of Laurent Schwartz

## Sur un théorème de Laurent Schwartz

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## A R T I C L E IN F O

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#### Abstract

We give a proof of a theorem of Schwartz on Borel graphs for linear transforms between Banach spaces, completely different from the original one. © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

\section*{R É S U M É}


Nous donnons une démonstration du théorème de Schwartz sur les graphes de Borel pour les transformées linéaires entre espaces de Banach, entièrement différente de l'originale.
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In [5], L. Schwartz proved for a large class of topological vector spaces that a linear map is continuous if its graph is a Borel set, and shortly thereafter, A. Martineau [3] and [4] showed that Schwartz's argument could be simplified. Both Schwartz and Martineau base their proofs on quite sophisticated applications of descriptive set theory: Souslin spaces, meager sets, and the like. A detailed account of Martineau's ideas can be found in the appendix to F. Treves book [6]. A further simplification and extension was given by N. Hogbe-Nlend in [2]. The purpose of this note is to provide a simple proof of Schwartz's result in the setting of Banach spaces.

Given a real Banach space $E$, a centered, Gaussian measure on $E$ is a Borel probability measure $\mu$ with the property that the pair

$$
\left(\frac{x_{1}+x_{2}}{\sqrt{2}}, \frac{x_{1}-x_{2}}{\sqrt{2}}\right)
$$

has the same distribution under $\mu^{2}$ as the pair $\left(x_{1}, x_{2}\right)$. When $E$ is separable, an equivalent statement is that, for each $x^{*} \in E^{*}$, the distribution $x \in E \mapsto\left\langle x, x^{*}\right\rangle \in \mathbb{R}$ is a centered Gaussian on $\mathbb{R}$. A renowned theorem of X . Fernique [1] guarantees that if $\mu$ is a centered, Gaussian measure on $E$, then there is an $\alpha>0$ for which $\int e^{\alpha\|x\|_{E}^{2}} \mu(\mathrm{~d} x)<\infty$. In particular, $\int\|x\|_{E}^{2} \mu(\mathrm{~d} x)<\infty$.

One way to construct centered, Gaussian measures on $E$ is to start with a sequence $\left\{x_{n}: n \geqslant 1\right\} \subseteq E$ with the property that $\sum_{n=1}^{\infty}\left\|x_{n}\right\|_{E}<\infty$. Next, set $\Omega=\mathbb{R}^{\mathbb{Z}^{+}}$with the product topology, and take $\mathbb{P}=\gamma^{\mathbb{Z}^{+}}$, where $\gamma$ is the standard Gauss measure on $\mathbb{R}$ (i.e., the one with mean 0 and variance 1 ). Then

$$
\mathbb{E}^{\mathbb{P}}\left[\sum_{n=1}^{\infty}\left|\omega_{n}\right|\left\|x_{n}\right\|_{E}\right]=\sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty}\left\|x_{n}\right\|_{E}<\infty
$$

[^0]and therefore there is a random variable $X$ such that $X(\omega)=\sum_{n=1}^{\infty} \omega_{n} x_{n}$ is $\mathbb{P}$-almost surely convergent in $E$. Furthermore, it is an easy matter to check that the distribution of $X$ is centered Gaussian.

Theorem. Let $E$ and $F$ be a pair of real Banach spaces and $\psi: E \rightarrow F$ a linear map. If $\psi$ is measurable with respect to every centered Gaussian measure on $E$, then $\psi$ is continuous.

Proof. Begin by observing that for every centered Gaussian measure $\mu$ on $E$, the distribution $v$ of $x \in E \mapsto \psi(x) \in F$ is a centered Gaussian measure on $F$. Indeed, because $\mu$ is centered Gaussian, ( $\psi\left(x_{1}\right), \psi\left(x_{2}\right)$ ) has the same distribution under $\mu^{2}$ as

$$
\left(\psi\left(\frac{x_{1}+x_{2}}{\sqrt{2}}\right), \psi\left(\frac{x_{1}-x_{2}}{\sqrt{2}}\right)\right)=\left(\frac{\psi\left(x_{1}\right)+\psi\left(x_{2}\right)}{\sqrt{2}}, \frac{\psi\left(x_{1}\right)-\psi\left(x_{2}\right)}{\sqrt{2}}\right) .
$$

As a consequence of the preceding, Fernique's theorem says that

$$
\int_{E}\|\psi(x)\|_{F}^{2} \mu(\mathrm{~d} x)=\int_{F}\|y\|_{F}^{2} v(\mathrm{~d} y)<\infty
$$

for every centered Gaussian measure $\mu$ on $E$.
Now suppose that $\psi$ is not continuous. Then there exists a sequence $\left\{x_{n}: n \geqslant 1\right\} \subseteq E$ and a sequence $\left\{y_{n}^{*}: n \geqslant 1\right\} \subseteq F^{*}$ such that $\left\|x_{n}\right\|_{E}=\frac{1}{n^{2}},\left\|y_{n}^{*}\right\|_{F}=1$, and $\left\langle\psi\left(x_{n}\right), y_{n}^{*}\right\rangle \geqslant n$. Referring to the construction given above, let $\mu$ be the distribution of the random variable $X$ corresponding to $\left\{x_{n}: n \geqslant 1\right\}$. At the same time, for each $m \geqslant 1$, let $X_{m}=\omega_{m} x_{m}$ and $X^{m}=X-X_{m}$. Then $X_{m}$ is independent of $X^{m}$ and the distributions of both are centered Gaussian. Hence, since $\psi(X)=\psi\left(X_{m}\right)+\psi\left(X^{m}\right)$,

$$
\infty>\int_{E}\|\psi(x)\|_{F}^{2} \mu(\mathrm{~d} x) \geqslant \int_{E}\left\langle\psi(x), y_{m}^{*}\right\rangle^{2} \mu(\mathrm{~d} x)=\mathbb{E}^{\mathbb{P}}\left[\left\langle\psi\left(X_{m}\right), y_{m}^{*}\right\rangle^{2}\right]+\mathbb{E}^{\mathbb{P}}\left[\left\langle\psi\left(X^{m}\right), y_{m}^{*}\right\rangle^{2}\right] \geqslant m^{2}
$$

which is obviously impossible.
As G. Pisier pointed out to me, at least when $\psi$ is Borel measurable, this theorem is an immediate consequence of Schwartz's. To check that it is, define $\Psi: E \times F \rightarrow F \times F$ by $\Psi(x, y)=(\psi(x), y)$. If $\psi$ is Borel measurable, so is $\Psi$. Furthermore, the graph of $\psi$ is the inverse image under $\psi$ of the diagonal in $F \times F$. Hence, the graph of $\psi$ is Borel measurable if $\psi$ is Borel measurable. To see that Schwartz's theorem for separable Banach spaces follows from the preceding theorem, assume that the graph $G$ of $\psi$ is Borel measurable. Then, for each Borel measurable $\Gamma \subseteq E,\{x: \psi(x) \in \Gamma\}$ is the image under the natural projection $\Pi_{E}: E \times F \rightarrow E$ of the Borel set $G \cap(\Gamma \times F)$, and, as such, is an analytic subset of $E$. Since analytic sets are measurable with respect to every Borel measure, it follows that $\psi$ is measurable with respect to all centered, Gaussian measures on $E$.

It is not clear to me just how tied to the Banach space setting the preceding proof is. Without too much trouble, one can extend it to the case when $E$ and $F$ are a countably normed Fréchet spaces. However, it seems that to go further will require a new idea.

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