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# On a theorem of Laurent Schwartz

### Sur un théorème de Laurent Schwartz

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ARTICLE INFO	ABSTRACT
Article history: Received 5 November 2010 Accepted 8 November 2010 Available online 22 December 2010	We give a proof of a theorem of Schwartz on Borel graphs for linear transforms between Banach spaces, completely different from the original one. © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
Presented by Michel Talagrand	R É S U M É
	Nous donnons une démonstration du théorème de Schwartz sur les graphes de Borel pour les transformées linéaires entre espaces de Banach, entièrement différente de l'originale. © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

In [5], L. Schwartz proved for a large class of topological vector spaces that a linear map is continuous if its graph is a Borel set, and shortly thereafter, A. Martineau [3] and [4] showed that Schwartz's argument could be simplified. Both Schwartz and Martineau base their proofs on quite sophisticated applications of descriptive set theory: Souslin spaces, meager sets, and the like. A detailed account of Martineau's ideas can be found in the appendix to F. Treves book [6]. A further simplification and extension was given by N. Hogbe-Nlend in [2]. The purpose of this note is to provide a simple proof of Schwartz's result in the setting of Banach spaces.

Given a real Banach space E, a centered, Gaussian measure on E is a Borel probability measure  $\mu$  with the property that the pair

$$\left(\frac{x_1+x_2}{\sqrt{2}},\frac{x_1-x_2}{\sqrt{2}}\right)$$

has the same distribution under  $\mu^2$  as the pair  $(x_1, x_2)$ . When *E* is separable, an equivalent statement is that, for each  $x^* \in E^*$ , the distribution  $x \in E \mapsto \langle x, x^* \rangle \in \mathbb{R}$  is a centered Gaussian on  $\mathbb{R}$ . A renowned theorem of X. Fernique [1] guarantees that if  $\mu$  is a centered, Gaussian measure on *E*, then there is an  $\alpha > 0$  for which  $\int e^{\alpha \|x\|_E^2} \mu(dx) < \infty$ . In particular,  $\int \|x\|_E^2 \mu(dx) < \infty$ .

One way to construct centered, Gaussian measures on *E* is to start with a sequence  $\{x_n: n \ge 1\} \subseteq E$  with the property that  $\sum_{n=1}^{\infty} ||x_n||_E < \infty$ . Next, set  $\Omega = \mathbb{R}^{\mathbb{Z}^+}$  with the product topology, and take  $\mathbb{P} = \gamma^{\mathbb{Z}^+}$ , where  $\gamma$  is the standard Gauss measure on  $\mathbb{R}$  (i.e., the one with mean 0 and variance 1). Then

$$\mathbb{E}^{\mathbb{P}}\left[\sum_{n=1}^{\infty}|\omega_n|\|x_n\|_E\right] = \sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\|x_n\|_E < \infty,$$

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and therefore there is a random variable *X* such that  $X(\omega) = \sum_{n=1}^{\infty} \omega_n x_n$  is  $\mathbb{P}$ -almost surely convergent in *E*. Furthermore, it is an easy matter to check that the distribution of *X* is centered Gaussian.

**Theorem.** Let *E* and *F* be a pair of real Banach spaces and  $\psi : E \to F$  a linear map. If  $\psi$  is measurable with respect to every centered Gaussian measure on *E*, then  $\psi$  is continuous.

**Proof.** Begin by observing that for every centered Gaussian measure  $\mu$  on E, the distribution  $\nu$  of  $x \in E \mapsto \psi(x) \in F$  is a centered Gaussian measure on F. Indeed, because  $\mu$  is centered Gaussian,  $(\psi(x_1), \psi(x_2))$  has the same distribution under  $\mu^2$  as

$$\left(\psi\left(\frac{x_1+x_2}{\sqrt{2}}\right),\psi\left(\frac{x_1-x_2}{\sqrt{2}}\right)\right) = \left(\frac{\psi(x_1)+\psi(x_2)}{\sqrt{2}},\frac{\psi(x_1)-\psi(x_2)}{\sqrt{2}}\right).$$

As a consequence of the preceding, Fernique's theorem says that

$$\int_{E} \|\psi(x)\|_{F}^{2} \mu(dx) = \int_{F} \|y\|_{F}^{2} \nu(dy) < \infty$$

for every centered Gaussian measure  $\mu$  on *E*.

Now suppose that  $\psi$  is not continuous. Then there exists a sequence  $\{x_n: n \ge 1\} \subseteq E$  and a sequence  $\{y_n^*: n \ge 1\} \subseteq F^*$ such that  $\|x_n\|_E = \frac{1}{n^2}$ ,  $\|y_n^*\|_F = 1$ , and  $\langle \psi(x_n), y_n^* \rangle \ge n$ . Referring to the construction given above, let  $\mu$  be the distribution of the random variable X corresponding to  $\{x_n: n \ge 1\}$ . At the same time, for each  $m \ge 1$ , let  $X_m = \omega_m x_m$  and  $X^m = X - X_m$ . Then  $X_m$  is independent of  $X^m$  and the distributions of both are centered Gaussian. Hence, since  $\psi(X) = \psi(X_m) + \psi(X^m)$ ,

$$\infty > \int_{E} \left\| \psi(x) \right\|_{F}^{2} \mu(\mathrm{d}x) \ge \int_{E} \left\langle \psi(x), y_{m}^{*} \right\rangle^{2} \mu(\mathrm{d}x) = \mathbb{E}^{\mathbb{P}} \left[ \left\langle \psi(X_{m}), y_{m}^{*} \right\rangle^{2} \right] + \mathbb{E}^{\mathbb{P}} \left[ \left\langle \psi(X^{m}), y_{m}^{*} \right\rangle^{2} \right] \ge m^{2}$$

which is obviously impossible.  $\Box$ 

As G. Pisier pointed out to me, at least when  $\psi$  is Borel measurable, this theorem is an immediate consequence of Schwartz's. To check that it is, define  $\Psi : E \times F \to F \times F$  by  $\Psi(x, y) = (\psi(x), y)$ . If  $\psi$  is Borel measurable, so is  $\Psi$ . Furthermore, the graph of  $\psi$  is the inverse image under  $\Psi$  of the diagonal in  $F \times F$ . Hence, the graph of  $\psi$  is Borel measurable if  $\psi$  is Borel measurable. To see that Schwartz's theorem for separable Banach spaces follows from the preceding theorem, assume that the graph G of  $\psi$  is Borel measurable. Then, for each Borel measurable  $\Gamma \subseteq E$ ,  $\{x: \psi(x) \in \Gamma\}$  is the image under the natural projection  $\Pi_E : E \times F \to E$  of the Borel set  $G \cap (\Gamma \times F)$ , and, as such, is an analytic subset of E. Since analytic sets are measurable with respect to every Borel measure, it follows that  $\psi$  is measurable with respect to all centered, Gaussian measures on E.

It is not clear to me just how tied to the Banach space setting the preceding proof is. Without too much trouble, one can extend it to the case when E and F are a countably normed Fréchet spaces. However, it seems that to go further will require a new idea.

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