In [5], L. Schwartz proved for a large class of topological vector spaces that a linear map is continuous if its graph is a Borel set, and shortly thereafter, A. Martineau [3] and [4] showed that Schwartz’s argument could be simplified. Both Schwartz and Martineau base their proofs on quite sophisticated applications of descriptive set theory: Souslin spaces, meager sets, and the like. A detailed account of Martineau’s ideas can be found in the appendix to F. Treves book [6]. A further simplification and extension was given by N. Hogbe-Nlend in [2]. The purpose of this note is to provide a simple proof of Schwartz’s result in the setting of Banach spaces.

Given a real Banach space $E$, a centered, Gaussian measure on $E$ is a Borel probability measure $\mu$ with the property that
\[
\left( \frac{x_1 + x_2}{\sqrt{2}}, \frac{x_1 - x_2}{\sqrt{2}} \right)
\]
has the same distribution under $\mu^2$ as the pair $(x_1, x_2)$. When $E$ is separable, an equivalent statement is that, for each $x^* \in E^*$, the distribution $x \in E \mapsto \langle x, x^* \rangle \in \mathbb{R}$ is a centered Gaussian on $\mathbb{R}$. A renowned theorem of X. Fernique [1] guarantees that if $\mu$ is a centered, Gaussian measure on $E$, then there is an $\alpha > 0$ for which $\int e^{\alpha \|x\|_E^2} \mu(dx) < \infty$. In particular, $\int \|x\|_E^2 \mu(dx) < \infty$.

One way to construct centered, Gaussian measures on $E$ is to start with a sequence $\{x_n; \ n \geq 1\} \subseteq E$ with the property that $\sum_{n=1}^{\infty} \|x_n\|_E < \infty$. Next, set $\Omega = \mathbb{R}^{\mathbb{Z}^+}$ with the product topology, and take $P = \gamma^{\mathbb{Z}^+}$, where $\gamma$ is the standard Gauss measure on $\mathbb{R}$ (i.e., the one with mean 0 and variance 1). Then
\[
\mathbb{E}^P \left[ \sum_{n=1}^{\infty} (\omega_n \|x_n\|_E) \right] = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \|x_n\|_E < \infty.
\]
and therefore there is a random variable $X$ such that $X(\omega) = \sum_{n=1}^{\infty} \omega_n x_n$ is $\mathbb{P}$-almost surely convergent in $E$. Furthermore, it is an easy matter to check that the distribution of $X$ is centered Gaussian.

**Theorem.** Let $E$ and $F$ be a pair of real Banach spaces and $\psi : E \to F$ a linear map. If $\psi$ is measurable with respect to every centered Gaussian measure on $E$, then $\psi$ is continuous.

**Proof.** Begin by observing that for every centered Gaussian measure $\mu$ on $E$, the distribution $\nu$ of $x \in E \mapsto \psi(x) \in F$ is a centered Gaussian measure on $F$. Indeed, because $\mu$ is centered Gaussian, $(\psi(x_1), \psi(x_2))$ has the same distribution under $\mu^2$ as

$$
\left( \psi \left( \frac{x_1 + x_2}{\sqrt{2}} \right), \psi \left( \frac{x_1 - x_2}{\sqrt{2}} \right) \right)
= \left( \frac{\psi(x_1) + \psi(x_2)}{\sqrt{2}}, \frac{\psi(x_1) - \psi(x_2)}{\sqrt{2}} \right).
$$

As a consequence of the preceding, Fernique’s theorem says that

$$
\int_E \|\psi(x)\|_F^2 \mu(dx) = \int_F \|y\|_F^2 \nu(dy) < \infty
$$

for every centered Gaussian measure $\mu$ on $E$.

Now suppose that $\psi$ is not continuous. Then there exists a sequence $\{x_n : n \geq 1\} \subseteq E$ and a sequence $\{y_n^* : n \geq 1\} \subseteq F^*$ such that $\|x_n\|_E = \frac{1}{n}$, $\|y_n^*\|_F = 1$, and $\langle \psi(x_n), y_n^* \rangle \geq n$. Referring to the construction given above, let $\mu$ be the distribution of the random variable $X$ corresponding to $\{x_n : n \geq 1\}$. At the same time, for each $m \geq 1$, let $X_m = \omega_m x_m$ and $X_m = X - x_m$. Then $X_m$ is independent of $X^n$ and the distributions of both are centered Gaussian. Hence, since $\psi(X) = \psi(X_m) + \psi(X^n)$,

$$
\infty > \int_E \|\psi(x)\|_F^2 \mu(dx) \geq \int_E \langle \psi(x), y_m^* \rangle^2 \mu(dx) = \mathbb{E}^E[\langle \psi(X_m), y_m^* \rangle^2] + \mathbb{E}^F[\langle \psi(X^n), y_m^* \rangle^2] \geq m^2,
$$

which is obviously impossible. □

As G. Pisier pointed out to me, at least when $\psi$ is Borel measurable, this theorem is an immediate consequence of Schwartz’s. To check that it is, define $\Psi : E \times F \to F \times F$ by $\Psi(x, y) = (\psi(x), y)$. If $\psi$ is Borel measurable, so is $\Psi$. Furthermore, the graph $G$ of $\psi$ is the inverse image under $\Psi$ of the diagonal in $F \times F$. Hence, the graph of $\Psi$ is Borel measurable if $\psi$ is Borel measurable. To see that Schwartz’s theorem for separable Banach spaces follows from the preceding theorem, assume that the graph $G$ of $\psi$ is Borel measurable. Then, for each Borel measurable $\Gamma \subseteq E$, $(x : \psi(x) \in \Gamma)$ is the image under the natural projection $\Pi_E : E \times F \to E$ of the Borel set $G \cap (\Gamma \times F)$, and, as such, is an analytic subset of $E$. Since analytic sets are measurable with respect to every Borel measure, it follows that $\psi$ is measurable with respect to all centered Gaussian measures on $E$.

It is not clear to me just how tied to the Banach space setting the preceding proof is. Without too much trouble, one can extend it to the case when $E$ and $F$ are a countably normed Fréchet spaces. However, it seems that to go further will require a new idea.

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**References**


