Dynamical Systems

A model for the parabolic slices Per$_1(e^{2\pi ip/q})$ in moduli space of quadratic rational maps

Un modèle pour les sections paraboliques Per$_1(e^{2\pi ip/q})$ de l'espace des modules des fractions rationnelles quadratiques

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1. Introduction

Let $\mathcal{M}_2$ denote the moduli space of Möbius conjugacy classes of quadratic rational maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Following definitions and statements from Milnor [3], consider loci:

$$\text{Per}_1(\lambda) = \{ [f] \in \mathcal{M}_2 : \text{f has a fixed point with eigenvalue } \lambda \} \cong \mathbb{C}. $$

Here the focus will be on parabolic slices $\text{Per}_1(\omega)$, with $\omega = e^{2\pi ip/q}$, $p/q \neq 0/1$, i.e. those consisting of equivalence classes of maps with a parabolic fixed point with eigenvalue $\omega$. In such a slice the dynamics is characterized according to the behavior of the critical points. The relatedness locus $\mathcal{R}_{\omega}$ in $\text{Per}_1(\omega)$ is defined by:

$$\mathcal{R}_{\omega} = \{ [f] \in \text{Per}_1(\omega) : \lim_{n \to \infty} f^n(c_1) = z_0 = \lim_{n \to \infty} f^n(c_2) \},$$

where $z_0$ is the (persistent) parabolic fixed point and $c_1$ and $c_2$ are the critical points of $f$. The locus $\mathcal{R}_{\omega}$ is neither open nor closed. It consists of open, connected components of maps where both critical points are in the parabolic basin.
(in [3] — for hyperbolic components — called bitransitive and capture components respectively, according to whether both or only one critical point is in the immediate basin, also studied by Rees [6] under different names), a countable set of points corresponding to maps where one critical point is eventually mapped to the parabolic fixed point, and a finite set of points corresponding to maps where the parabolic fixed point is degenerate, i.e. has two q-cycles of components in the immediate basin. In the slice Per1(0) the relatedness locus $\mathcal{R}^0$ is the escape locus $\mathbb{C} \setminus M$, where $M$ is the Mandelbrot set, the connectedness locus in the slice of polynomials.

2. The model

The objective is to construct a model for $X^\omega$ (see Theorem 3.1). Consider the quadratic polynomial

$$P_\omega(z) = \omega z + z^2,$$

with a parabolic fixed point with multiplier $\omega$ at 0. This fixed point is called the $\alpha$-fixed point. Let $\Lambda_\omega$ denote the parabolic basin of 0 for $P_\omega$ and define also an augmented basin $\tilde{\Lambda}_\omega$:

$$\tilde{\Lambda}_\omega = \left\{ z \in \mathbb{C} : \lim_{n \to \infty} P_\omega^n(z) = 0 \right\} = \Lambda_\omega \cup \left\{ z \in \mathbb{C} : \exists n \geq 0, \ P_\omega^n(z) = 0 \right\}.$$  

The immediate basin has $q$ components, labelled $B_j$, $j \in \{0, \ldots, q - 1\}$ counter-clockwise, so that $B_0$ contains the critical point $-\omega/2$. It follows from the theory of quadratic polynomials that there are $q$ external rays landing at 0, dividing $\hat{\mathbb{C}}$ into $q$ components. Let $S_p$ denote the component containing the critical value $P_\omega(-\omega/2)$. Let $\phi_\omega : \Lambda_\omega \to \mathbb{C}$ be an extended Fatou coordinate for $P_\omega^q$, i.e. a surjective holomorphic map, of infinite degree, with critical points at the critical point $-\omega/2$ of $P_\omega$ and at all its pre-images, so that $\phi_\omega \circ P_\omega^q = 1 + \phi_\omega$.

Normalize $\phi_\omega$ so that $\phi_\omega \circ P_\omega = 1/q + \phi_\omega$ and $\phi_\omega(-\omega/2) = 0$. Let $P^j_\delta \subset B_j$ be the connected component of $\phi_\omega^{-1}((z = x + iy : x > \delta)) \cap B_j$ with the fixed point 0 on its boundary, called a petal.

Denote by $X^\omega$ the subset of the Riemann sphere obtained by removing the union of the closures of the petals $P^j_q$.

$$X^\omega = \mathbb{C} \setminus \bigcup_{j=0}^{q-1} P^j_{1/q}.$$ Let $\hat{X}^\omega = \hat{\mathbb{C}} \setminus \mathbb{D}$ be the Carathéodory compactification of $X^\omega$, i.e. the disjoint union of $X^\omega$ and the set consisting of all prime ends of $X^\omega$. The boundary of $\hat{X}^\omega$ can be naturally identified with the boundaries of the petals, together with $q$ copies of the $\alpha$-fixed point, corresponding to the $q$ different accesses to the $\alpha$-fixed point from $X^\omega$. The copies are labelled $\hat{\alpha}_j$, $j \in \{0, \ldots, q - 1\}$ counter-clockwise, so that $\hat{\alpha}_j$ is an endpoint of $\partial P^j$ and $\partial P^{j+1}$ (Fig. 1). For the remainder of this section sets in $\hat{\mathbb{C}} \setminus \bigcup_{j=0}^{q-1} P^j_{1/q}$ are to be understood as subsets of $\hat{X}^\omega$, i.e. with $q$ copies of the $\alpha$-fixed point.

Now an equivalence relation on $\hat{X}^\omega$ is defined. To shorten notation let $\partial P^j = \partial P^j_{1/q}$.

**Definition 2.1.** Two points $z_1 \in \partial P^j$ and $z_2 \in \partial P^k$ are called equivalent modulo $p/q$, written $z_1 \sim_{p/q} z_2$, if the following two conditions are satisfied:

- $j + k = 2p \mod q$, and
- $\phi_\omega(z_1) + \phi_\omega(z_2) = 2/q$.

Two points $\hat{\alpha}_j$ and $\hat{\alpha}_k$ are said to be equivalent modulo $p/q$ if $j + k = (2p - 1) \mod q$.

Let $X^\omega$ be the quotient of $\hat{X}^\omega$ under the equivalence relation $\sim_{p/q}$, let $\pi_\omega : \hat{X}^\omega \to X^\omega$ denote the projection map induced by $\sim_{p/q}$ and let $\nu_\omega = \pi_\omega(\partial \hat{X}^\omega) \subset X^\omega$ denote the scar after gluing the real-analytic boundaries of the petals back together.
under \( \sim_{p/q} \). It can be proved [7] that the map \( \pi_{\omega} \) gives \( X^\omega \) a Riemann surface structure which extends the initial structure of \( X^\omega \), and so that \( X^\omega \cong \hat{C} \).

**Definition 2.2.** The model space \( \hat{A}^\omega \subset X^\omega \) for \( R^\omega \) is defined by \( \hat{A}^\omega = (\tilde{A}^\omega \setminus (S_p \cup \bigcup_{j=0}^{q-1} P_{1/q}^j)) / \sim_{p/q} \).

The set \( S_p \) is removed because it in some sense corresponds to non-realizable matings (i.e. matings of \( P_\omega \) with maps from the conjugate limb \( L_{-p/q} \)).

**Definition 2.3.** From the Fatou coordinate define a tree in \( \hat{A}^\omega \), called a bubble-tree and denoted \( \hat{T}^\omega \), by:

\[
\hat{T}^\omega = \pi_\omega \left( \varphi^{-1}_\omega(\mathbb{R}) \cup \bigcup_{n>0} P^{-n}_\omega(0) \right) \cup V_\omega.
\]

The bubble-tree has vertices at pre-fixed and (pre)-critical points of \( P_\omega \) and at the points \( \pi_{\omega}(\alpha_i), i \in \{0, \ldots, q-1\} \). A metric is defined on the tree by assigning length one to every edge and letting the distance between any two vertices be the sum of the lengths of the edges in the unique finite path between them.

### 3. Faithfulness of the model

**Theorem 3.1.** There exists a bijective map \( \chi^\omega : R^\omega \to \hat{A}^\omega \), which is conformal in \( \text{int}(R^\omega) \). The inverse \( (\chi^\omega)^{-1} \) is continuous on compact subsets of the bubble-tree \( \hat{T}^\omega \), with respect to the topology induced by the metric on the tree.

The inverse \( (\chi^\omega)^{-1} \) does not extend to \( \partial \hat{A}^\omega \) as an injective map, but it seems tempting to conjecture that it extends to \( \partial \hat{A}^\omega \) as a continuous, surjective map. However, one would expect a proof of continuity to be similar to proving local connectivity of \( M \). A more accessible conjecture would be that \( (\chi^\omega)^{-1} \) extends continuously to points in \( \partial \hat{A}^\omega \) that correspond to (pre)-periodic points in \( \partial A_\omega \) (the Julia set for \( P_\omega \)) and to the boundary of components that correspond to strictly pre-periodic components of \( A_\omega \).

Let \( f_\sigma \in R^\omega \) be non-degenerate parabolic, let \( \phi_\sigma \) be a Fatou coordinate for \( f_\sigma \) and let \( R \in \mathbb{R} \) be smallest so that the union of petals \( \bigcup_{j=0}^{q-1} P^j_{\sigma,R} \) contains no critical point, but contains (at least) one critical point on the boundary. These petals are called maximal attracting petals and (one of) the critical point(s) on the boundary is called the closest critical point and denoted \( c_1 \). The other critical point is then called the second critical point and denoted \( c_2 \). The critical values under \( f_\sigma \) are denoted \( v_1 \) and \( v_2 \) respectively. Normalize the Fatou coordinate \( \phi_\sigma \) so that \( \phi_\sigma(c_1) = 0 \) and \( \phi_\sigma \circ f_\sigma = 1/q + \phi_\sigma \). Let \( U_0^\sigma = \bigcup_{j=0}^{q-1} P^j_\sigma \) be the maximal attracting petals for \( P_\omega \), and \( U_0^\sigma = \bigcup_{j=0}^{q-1} P^j_\sigma \) the maximal attracting petals for \( f_\sigma \). Further, let \( U_0^\alpha = f^{-1}_\alpha(U_0^\sigma) \) and \( U_0^\beta = f^{-1}_\beta(U_0^\sigma) \). The map \( \chi^\omega \) is constructed via a dynamical conjugacy:

**Lemma 3.2.** For all non-degenerate parabolic \( f_\sigma \in R^\omega \) there exists a continuous conjugacy \( \eta_{\sigma,\omega} : \overline{U_\sigma^\alpha} \to \tilde{A}_\omega \) between \( f_\sigma \) and \( P_\omega \), so that \( \eta_{\sigma,\omega}(P^j_{\sigma,\omega}) = P^j_0 \) for \( j \in \{0, \ldots, q-1\} \). The domain \( U_\sigma = U_0^\sigma \) for some \( n \in \mathbb{N} \cup \{0\} \) and \( \overline{U_\sigma} \) contains both critical values \( v_1 \) and \( v_2 \). The conjugacy \( \eta_{\sigma,\omega} \) is holomorphic in \( U_\sigma \).

**Proof.** Let \( f_\sigma \in R^\omega \). Recall that \( \phi_\omega \) and \( \phi_\sigma \) are Fatou coordinates for \( P_\omega \) and \( f_\sigma \) respectively. The map \( \eta_{\sigma,\omega} = \varphi^{-1}_\omega \circ \phi_\sigma : \overline{U_0^\sigma} \to \overline{U_0^\omega} \), constructed so that \( \eta_{\sigma,\omega}(P^j_{\sigma,\omega}) = P^j_0 \) for all \( j \in \{0, \ldots, q-1\} \), is a homeomorphism, conformal in \( U_0^\sigma \) and it conjugates \( f_\sigma \) to \( P_\omega \). If \( v_2 \in \overline{U_0^\sigma} \), then \( U_\sigma = U_0^\omega \) and the proof is done. If not, there exists \( N > 0 \) so that \( v_2 \in \overline{U_0^\sigma} \setminus \overline{U_0^{N-1}} \) and the conjugacy extends, by iterated lifting with respect to the dynamics, to a conjugacy \( \eta_{\sigma,\omega} : \overline{U_\sigma^N} \to \overline{U_\omega^N} \). Each lift is chosen to agree with the previous map on their common domain of definition.

**Lemma 3.3.** For all non-degenerate parabolic \( f_\sigma \in R^\omega, \eta_{\sigma,\omega}(v_2) \in \tilde{A}_\omega \setminus S_p \).

**Sketch of Proof.** The proof is by contradiction. Assume \( \exists f_\sigma \in R^\omega \) so that \( \eta_{\sigma,\omega}(v_2) \in \tilde{A}_\omega \cap S_p \). Let \( \overline{U} = \overline{U_0^\sigma} \) be the maximal domain of the conjugacy \( \eta_{\sigma,\omega} \), so that \( v_1, v_2 \in \overline{U} \), and \( \overline{V} = \tilde{C} \setminus \overline{U} \subset \mathbb{D} \). Hence \( f_\sigma \) has two univalent inverse branches \( f^{-1}_\sigma : \overline{V} \to \overline{V} \). Let \( \mathcal{P} \) denote the union of \( q \) repelling petals at the parabolic fixed point \( z_0 \), sufficiently small so that \( \mathcal{P} \subset f^{-1}_\sigma(V) \subset \overline{V} \). If \( f_\sigma \in R^\omega \) so that \( \eta_{\sigma,\omega}(v_2) \in \tilde{A}_\omega \cap S_p \), then \( f^{-1}_\sigma(\overline{U}) \) separates \( \alpha \) from its co-preimage \( \alpha' \), and \( \mathcal{P} \) is then contained in the image of one of the inverse branches \( f^{-1}_\sigma : \overline{V} \to \overline{V} \). But then this inverse branch has an attracting \( q \)-cycle on the ideal boundary, contradicting the Denjoy–Wolff theorem.

**Strategy of Proof of Theorem 3.1.** Definition of the map \( \chi^\omega \). Let \( f_\sigma \in R^\omega \), with second critical value \( v_2 \). If \( f_\sigma \) is degenerate parabolic, then it has two \( q \)-cycles of components as immediate basin, with each cycle containing a critical point. In this case choose one of the critical points to be the closest critical point \( c_1 \), and name the components in the corresponding cycle
$B_0, \ldots, B_{q-1}$ counter-clockwise, so that $B_0$ contains the critical point $c_1$. The components of the other $q$-cycle will be named counter-clockwise so that the component $B_j'$ is the component between $B_j$ and $B_{(j+1) \mod q}$. The map $\chi^\omega : \mathbb{R}^\omega \to \hat{\Lambda}^\omega$ is defined by:

\[
\chi^\omega(\sigma) = \begin{cases} 
\pi_\omega \circ \eta_{\sigma, \omega}(v_2) & \text{if } \sigma \text{ non-degenerate,} \\
\pi_\omega(\hat{\alpha}_j) & \text{if } \sigma \text{ degenerate and } v_2 \in B_j'.
\end{cases}
\]

The map is well defined by Lemmas 3.2 and 3.3, and by the equivalence relation $\sim_{p/q}$, which identifies the images $\eta_{\sigma, \omega}(v_2)$ and $\hat{\alpha}_j$'s respectively, in the cases where there is an ambiguity in the choice of $v_2$. That the map $\chi^\omega$ is holomorphic in the interior will follow from holomorphic dependence of the Fatou coordinate $\phi_\sigma$ on the parameter $\sigma$.

Injectivity follows by a classical pull-back argument, see for example [2] and [5], adapted to the parabolic situation. Surjectivity is proved by constructing a sequence of polynomial-like maps $f_n \in \text{Per}_1(\lambda_n)$, with $\lambda_n \in \mathbb{D}^*$, $\lambda_n \to \omega$ radially, so that the limiting map $f_\omega \in \text{Per}_1(\omega)$ has the correct position of the second critical value. This is done by using results from [1] on the escape loci in $\text{Per}_1(\lambda)$ and results on convergence of polynomial basins, built upon the star-construction from [4].

Continuity of the map $(\chi^\omega)^{-1}$ on compact subsets of the bubble-tree is proved by using that the map $\eta_{\sigma, \omega}$ preserves the combinatorial structure of the bubble-tree.

Following Wittner’s conjecture on the slice $\text{Per}_2(0)$ [8], and revising a folklore conjecture on parabolic parameter slices, the theorem leads to the conjecture that $\text{Per}_1(e^{2\pi ip/q})$ can be understood as the mating of $\hat{\Lambda}^\omega$ with a truncated Mandelbrot set, $M \setminus L_{-p/q}$, so that the bifurcation locus in $\text{Per}_1(e^{2\pi ip/q})$ is homeomorphic to the mating of $\partial \hat{\Lambda}^\omega$ with $\partial (M \setminus L_{-p/q})$.

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References

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