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# BV functions in a Gelfand triple and the stochastic reflection problem on a convex set of a Hilbert space $^{\updownarrow}$

## *Fonctions BV dans triplet de Gelfand et le problème de réflexion sur un ensemble convexe d'un espace de Hilbert*

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#### ABSTRACT

In this Note we introduce BV functions in a Gelfand triple, which is an extension of BV functions in Ambrosio et al., preprint [1], by using Dirichlet form theory. By this definition, we can consider the stochastic reflection problem associated with a self-adjoint operator A and a cylindrical Wiener process on a convex set  $\Gamma$ . We prove the existence and uniqueness of a strong solution of this problem when  $\Gamma$  is a regular convex set. The result is also extended to the non-symmetric case. Finally, we extend our results to the case when  $\Gamma = K_{\alpha}$ , where  $K_{\alpha} = \{f \in L^2(0, 1) \mid f \ge -\alpha\}, \alpha \ge 0$ .

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#### RÉSUMÉ

Dans cette Note, on introduit des fonctions BV dans un triplet de Gelfand qui est une extension de fonctions BV dans Ambrosio et al., preprint [1] en utilizant la forme de Dirichlet. Par cette définition, on peut considérer le problème de réflexion stochastique associé à un opérateur auto-adjoint *A* et un processus de Wiener cylindrique sur un ensemble convexe  $\Gamma$ . Nous démontrons l'existence et l'unicité d'une solution forte de ce problème si  $\Gamma$  et un ensemble convexe régulier. Le résultat est aussi étendu au cas non symétrique. Finalement, nous utilisons les fonctions BV dans le cas  $\Gamma = K_{\alpha}$ , où  $K_{\alpha} = \{f \in L^2(0, 1) \mid f \ge -\alpha\}, \alpha \ge 0.$ 

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#### 1. Dirichlet form and BV functions

Given a real separable Hilbert space H (with scalar product  $\langle \cdot, \cdot \rangle$  and norm denoted by  $|\cdot|$ ), assume that:

**Hypothesis 1.1.**  $A : D(A) \subset H \to H$  is a linear self-adjoint operator on H such that  $\langle Ax, x \rangle \ge \delta |x|^2, \forall x \in D(A)$ , for some  $\delta > 0$ . Moreover,  $A^{-1}$  is of trace class.  $\{e_j\}$  is an orthonormal basis in H consisting of eigen-functions for A, that is,  $Ae_j = \alpha_j e_j$ ,  $j \in \mathbb{N}$ , where  $\alpha_j \ge \delta$ .

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In the following  $D\varphi: H \to H$  is the Fréchet-derivative of a function  $\varphi: H \to \mathbb{R}$ . By  $C_b^1(H)$  we shall denote the set of all bounded differentiable functions with continuous and bounded derivatives. For  $K \subset H$ , the space  $C_b^1(K)$  is defined as the space of restrictions of all functions in  $C_b^1(H)$  to the subset K.  $\mu$  will denote the Gaussian measure in H with mean 0 and covariance operator  $Q := \frac{1}{2}A^{-1}$ . For  $\rho \in L_+^1(H, \mu)$ , we consider  $\mathcal{E}^{\rho}(u, v) = \frac{1}{2}\int_H \langle Du, Dv \rangle \rho(z)\mu(dz), u, v \in C_b^1(F)$ , where  $F = \text{Supp}[\rho \cdot \mu]$  and  $L_+^1(H, \mu)$  denotes the set of all non-negative elements in  $L^1(H, \mu)$ . Let QR(H) be the set of all functions  $\rho \in L_+^1(H, \mu)$  such that  $(\mathcal{E}^{\rho}, C_b^1(F))$  is closable on  $L^2(F; \rho \cdot \mu)$ . Its closure is denoted by  $(\mathcal{E}^{\rho}, \mathcal{F}^{\rho})$ .

**Theorem 1.2.** Let  $\rho \in QR(H)$ . Then  $(\mathcal{E}^{\rho}, \mathcal{F}^{\rho})$  is a quasi-regular local Dirichlet form on  $L^{2}(F; \rho \cdot \mu)$  in the sense of [6, IV, Definition 3.1].

By virtue of Theorem 1.2 and [6], there exists a diffusion process  $M^{\rho} = (X_t, P_z)$  on F associated with the Dirichlet form  $(\mathcal{E}^{\rho}, \mathcal{F}^{\rho})$ .  $M^{\rho}$  will be called distorted OU process on F. Since constant functions are in  $\mathcal{F}^{\rho}$  and  $\mathcal{E}^{\rho}(1, 1) = 0$ ,  $M^{\rho}$  is recurrent and conservative. Let  $A_{1/2}(x) := \int_0^x (\log(1+s))^{1/2} ds, x \ge 0$ , and let  $\psi$  be its complementary function, namely,  $\psi(y) := \int_0^y (A'_{1/2})^{-1}(t) dt = \int_0^y (\exp(t^2) - 1) dt$ . Define  $L(\log L)^{1/2} := \{f \mid A_{1/2}(\mid f \mid) \in L^1\}$ ,  $L^{\psi} := \{g \mid \psi(c \mid g \mid) \in L^1 \text{ for some } c > 0\}$  (cf. [7]). Let  $c_j, j \in \mathbb{N}$ , be a sequence in  $[1, \infty)$ . Define  $H_1 := \{x \in H \mid \sum_{j=1}^{\infty} \langle x, e_j \rangle^2 c_j^2 < \infty\}$ , equipped with the inner product  $\langle x, y \rangle_{H_1} := \sum_{j=1}^{\infty} c_j^2 \langle x, e_j \rangle \langle y, e_j \rangle$ . Then clearly  $(H_1, \langle , \rangle_{H_1})$  is a Hilbert space such that  $H_1 \subset H$  continuously and densely. Identifying H with its dual we obtain the continuous and dense embeddings  $H_1 \subset H (\equiv H^*) \subset H_1^*$ . It follows that  $H_1(z, v)_{H_1^*} = \langle z, v \rangle_H, \forall z \in H_1, v \in H$ , and that  $(H_1, H, H_1^*)$  is a Gelfand triple. We also introduce a family of H-valued function on H by

$$(C_b^1)_{D(A)\cap H_1} = \left\{ G: \ G(z) = \sum_{j=1}^m g_j(z) l^j, \ g_j \in C_b^1(H), \ l^j \in D(A) \cap H_1 \right\}.$$

Denote by  $D^*$  the adjoint of  $D: C_b^1(H) \subset L^2(H, \mu) \to L^2(H, \mu; H)$ . For  $\rho \in L(\log L)^{1/2}(H, \mu)$ , we put  $V(\rho) := \sup_{G \in (C_b^1)_{D(A) \cap H_1, \|G\|_{H_1} \leq 1} \int_H D^*G(z)\rho(z)\mu(dz)$ . A function  $\rho$  on H is called a BV function in the Gelfand triple  $(H_1, H, H_1^*)$  (denoted  $\rho \in BV(H, H_1)$  in notation), if  $\rho \in L(\log L)^{1/2}(H, \mu)$  and  $V(\rho)$  is finite. When  $H_1 = H = H_1^*$ , this coincides with the definition of BV functions defined in [1] and clearly  $BV(H, H) \subset BV(H, H_1)$ . This definition is a modification of BV function in abstract Wiener space introduced in [3] and [4].

**Theorem 1.3.** (i) Suppose  $\rho \in BV(H, H_1) \cap L^1_+(H, \mu)$ , then there exist a positive finite measure  $||d\rho||$  on H and a Borel-measurable map  $\sigma_{\rho} : H \to H^*_1$  such that  $||\sigma_{\rho}(z)||_{H^*_1} = 1 ||d\rho||$ -a.e.,  $V(\rho) = ||d\rho||(H)$ ,

$$\int_{H} D^{*}G(z)\rho(z)\mu(dz) = \int_{H} {}_{H_{1}} \langle G(z), \sigma_{\rho}(z) \rangle_{H_{1}^{*}} \|d\rho\|(dz), \quad \forall G \in (C_{b}^{1})_{D(A)\cap H_{1}}.$$
(1.1)

Further, if  $\rho \in QR(H)$ ,  $||d\rho||$  is  $\mathcal{E}^{\rho}$ -smooth, also,  $\sigma_{\rho}$  and  $||d\rho||$  are uniquely determined.

(ii) Conversely, if Eq. (1.1) holds for  $\rho \in L(\log L)^{1/2}(H, \mu)$  and for some positive finite measure  $||d\rho||$  and a map  $\sigma_{\rho}$  with the stated properties, then  $\rho \in BV(H, H_1)$  and  $V(\rho) = ||d\rho||(H)$ .

**Theorem 1.4.** Let  $\rho \in QR(H) \cap BV(H, H_1)$  and consider the measure  $||d\rho||$  and  $\sigma_\rho$  from Theorem 1.3(i). Then there is an  $\mathcal{E}^{\rho}$ -exceptional set  $S \subset F$  such that  $\forall z \in F \setminus S$ , under  $P_z$  there exists an  $\mathcal{M}_t$ -cylindrical Wiener process  $W^z$ , such that the sample paths of the associated distorted OU-process  $M^{\rho}$  on F satisfy the following: for  $l \in D(A) \cap H_1$ 

$$\langle l, X_t - X_0 \rangle = \int_0^t \langle l, dW_s^z \rangle + \frac{1}{2} \int_0^t H_1 \langle l, \sigma_\rho(X_s) \rangle_{H_1^*} dL_s^{\|d\rho\|} - \int_0^t \langle Al, X_s \rangle ds, \quad \forall t \ge 0, \ P_z \text{-}a.s.$$

Here  $L_t^{\|\mathbf{d}\rho\|}$  is the real valued PCAF associated with  $\|\mathbf{d}\rho\|$  by the Revuz correspondence.

#### 2. Reflected OU process

Consider the situation when  $\rho = I_{\Gamma}$ , the indicator of a set.

**Remark 2.1.** We emphasize that if  $\Gamma$  is a convex closed set in H, then for each  $z, l \in H$  the set  $\{s \in \mathbb{R} \mid z + sl \in \Gamma\}$  is a closed interval in  $\mathbb{R}$ , whose indicator function hence trivially has the Hamza property. Hence, in particular,  $I_{\Gamma} \in QR(H)$ .

#### 2.1. Reflected OU processes on regular convex set

Denote the corresponding objects  $\sigma_{\rho}$ ,  $\|dI_{\Gamma}\|$  in Theorem 1.3(i) by  $-\mathbf{n}_{\Gamma}$ ,  $\|\partial\Gamma\|$ , respectively.

**Hypothesis 2.1.1.** There exists a convex  $C^{\infty}$  function  $g: H \to R$  with g(0) = 0, g'(0) = 0, and  $D^2g$  strictly positively definite, that is,  $\langle D^2g(x)h, h \rangle \ge \gamma |h|^2$ ,  $\forall h \in H$ , where  $\gamma > 0$ , such that

$$\Gamma = \{ x \in H \colon g(x) \leq 1 \}, \qquad \partial \Gamma = \{ x \in H \colon g(x) = 1 \}$$

Moreover, we also suppose that  $D^2g$  is bounded on  $\Gamma$ . Finally, we also suppose that g and all its derivatives grow at infinity at most polynomially.

By using [2, Lemma 2.1], we have (1.1) for  $\rho = I_{\Gamma}$  with  $H = H_1$ . By the continuity property of surface measure given in [5], we have the following two theorems.

**Theorem 2.1.2.** Assume Hypothesis 2.1.1. Then  $I_{\Gamma} \in BV(H, H) \cap QR(H)$ .

**Theorem 2.1.3.** Assume Hypothesis 2.1.1. Then there exists an  $\mathcal{E}^{\rho}$ -exceptional set  $S \subset F$  such that  $\forall z \in F \setminus S$ , under  $P_z$  there exists an  $\mathcal{M}_t$ -cylindrical Wiener process  $W^z$ , such that the sample paths of the associated reflected OU-process  $M^{\rho}$  on F with  $\rho = I_{\Gamma}$  satisfy the following: for  $l \in D(A)$ 

$$\langle l, X_t - X_0 \rangle = \int_0^t \langle l, dW_s^z \rangle - \frac{1}{2} \int_0^t \langle l, \mathbf{n}_\Gamma(X_s) dL_s^{\|\partial\Gamma\|} \rangle - \int_0^t \langle Al, X_s \rangle \, \mathrm{d}s, \quad \forall t \ge 0, \ P_z \text{-}a.e.$$

where  $\mathbf{n}_{\Gamma} := \frac{Dg}{|Dg|}$  is the exterior normal to  $\Gamma$ , satisfying  $\langle \mathbf{n}_{\Gamma}(x), x - y \rangle \ge 0$ , for any  $y \in \Gamma$ ,  $x \in \partial \Gamma$  and  $\|\partial \Gamma\| = \mu_{\partial \Gamma}$ , where  $\mu_{\partial \Gamma}$  is the surface measure induced by  $\mu$  (cf. [2,5]).

Let  $\Gamma$  satisfy Hypothesis 2.1.1 and A satisfy Hypothesis 1.1. Consider the following stochastic differential inclusion in the Hilbert space H,

$$\begin{cases} dX(t) + (AX(t) + N_{\Gamma}(X(t))) dt \ni dW(t) \\ X(0) = x \end{cases}$$
(2.1)

where W(t) is a cylindrical Wiener process in H on a filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$  and  $N_{\Gamma}(x)$  is the normal cone to  $\Gamma$  at x.

**Definition 2.1.4.** A pair of continuous  $H \times R$  valued and  $\mathcal{F}_t$ -adapted processes  $(X(t), L(t)), t \in [0, T]$ , is called a solution of (2.1) if the following conditions hold:

- (i)  $X(t) \in \Gamma$ , *P*-a.s. for all  $t \in [0, T]$ ,
- (ii) *L* is an increasing process with the property  $\int_0^t I_{\partial\Gamma}(X_s(\omega)) dL_s(\omega) = L_t(\omega)$ ,  $t \ge 0$ , and we have for any  $l \in D(A)$ ,  $\langle l, X_t(\omega) - x \rangle = \langle l, W_t(\omega) - \int_0^t \mathbf{n}_{\Gamma}(X_s(\omega)) dL_s(\omega) \rangle - \langle Al, \int_0^t X_s(\omega) ds \rangle$  where  $\mathbf{n}_{\Gamma}$  is the exterior normal to  $\Gamma$ , satisfying  $\langle \mathbf{n}_{\Gamma}(x), x - y \rangle \ge 0$ ,  $\forall y \in \Gamma$ ,  $x \in \partial \Gamma$ .

**Theorem 2.1.5.** If  $\Gamma$  satisfies Hypothesis 2.1.1, then there exists M,  $I_{\Gamma} \cdot \mu(M) = 1$ , such that for every  $x \in M$ , (2.1) has a pathwise unique continuous strong solution in the sense of Definition 2.1.4. Moreover  $X(t) \in M$  for all  $t \ge 0$ ,  $P_x$ -a.s.

**Remark 2.1.6.** We can extend all these results to non-symmetric Dirichlet forms obtained by first order perturbation of the above Dirichlet form.

2.2. Reflection OU processes on a class of convex sets

Now we consider the case when  $H = L^2(0, 1)$ ,  $\rho = I_{K_\alpha}$ , where  $K_\alpha = \{f \in H \mid f \ge -\alpha\}$ ,  $\alpha \ge 0$ , and  $A = -\frac{1}{2}\frac{d^2}{dr^2}$  with Dirichlet boundary condition on [0, 1]. Take  $c_j = (j\pi)^{\frac{1}{2}+\varepsilon}$  if  $\alpha > 0$ ,  $c_j = (j\pi)^{\beta}$  if  $\alpha = 0$ , where  $\varepsilon \in (0, \frac{3}{2}]$  and  $\beta \in (\frac{3}{2}, 2]$  respectively. Then  $D(A) \subset H_1$  continuously for all  $\alpha \ge 0$ . By using [8, (1), (2)], we can prove the following theorem.

**Theorem 2.2.1.**  $I_{K_{\alpha}} \in BV(H, H_1) \cap QR(H)$ .

**Remark 2.2.2.** It has been proved by Guan Qingyang that  $I_{K_{\alpha}}$  is not in BV(H, H). Since we have Theorem 2.2.1, we denote the corresponding objects  $\sigma_{\rho}$ ,  $\|dI_{K_{\alpha}}\|$  in Theorem 1.3(i) by  $n_{\alpha}$ ,  $|\sigma_{\alpha}|$ , respectively.

**Theorem 2.2.3.** Let  $\rho = I_{K_{\alpha}}$ . Then there is an  $\mathcal{E}^{\rho}$ -exceptional set  $S \subset F$  such that  $\forall z \in F \setminus S$ , under  $P_z$  there exists an  $\mathcal{M}_t$ -cylindrical Wiener process  $W^z$ , such that the sample paths of the associated distorted OU-process  $M^{\rho}$  on F satisfy the following: for  $l \in D(A)$ 

$$\langle l, X_t - X_0 \rangle = \int_0^t \langle l, dW_s \rangle + \frac{1}{2} \int_0^t {}_{H_1} \langle l, n_\alpha(X_s) \rangle_{H_1^*} dL_s^{|\sigma_\alpha|} - \int_0^t \langle Al, X_s \rangle ds, \quad P_z\text{-}a.e.$$

Here,  $L_t^{|\sigma_{\alpha}|}(\omega)$  is a real valued PCAF associated with  $|\sigma_{\alpha}|$  by the Revuz correspondence, satisfying  $I_{\{X_s+\alpha\neq 0\}} dL_s^{|\sigma_{\alpha}|} = 0$ , and for every  $z \in F$ ,  $P_z[X_t \in C_0[0, 1]$  for a.e.  $t \in [0, \infty)] = 1$ .

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