Functional Analysis/Probability Theory

# BV functions in a Gelfand triple and the stochastic reflection problem on a convex set of a Hilbert space ${ }^{\text {为 }}$ 

# Fonctions BV dans triplet de Gelfand et le problème de réflexion sur un ensemble convexe d'un espace de Hilbert 

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#### Abstract

In this Note we introduce BV functions in a Gelfand triple, which is an extension of BV functions in Ambrosio et al., preprint [1], by using Dirichlet form theory. By this definition, we can consider the stochastic reflection problem associated with a self-adjoint operator $A$ and a cylindrical Wiener process on a convex set $\Gamma$. We prove the existence and uniqueness of a strong solution of this problem when $\Gamma$ is a regular convex set. The result is also extended to the non-symmetric case. Finally, we extend our results to the case when $\Gamma=K_{\alpha}$, where $K_{\alpha}=\left\{f \in L^{2}(0,1) \mid f \geqslant-\alpha\right\}, \alpha \geqslant 0$. © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

\section*{R É S U M É}


Dans cette Note, on introduit des fonctions BV dans un triplet de Gelfand qui est une extension de fonctions BV dans Ambrosio et al., preprint [1] en utilizant la forme de Dirichlet. Par cette définition, on peut considérer le problème de réflexion stochastique associé à un opérateur auto-adjoint $A$ et un processus de Wiener cylindrique sur un ensemble convexe $\Gamma$. Nous démontrons l'existence et l'unicité d'une solution forte de ce problème si $\Gamma$ et un ensemble convexe régulier. Le résultat est aussi étendu au cas non symétrique. Finalement, nous utilisons les fonctions BV dans le cas $\Gamma=K_{\alpha}$, où $K_{\alpha}=\left\{f \in L^{2}(0,1) \mid f \geqslant-\alpha\right\}, \alpha \geqslant 0$.
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## 1. Dirichlet form and BV functions

Given a real separable Hilbert space $H$ (with scalar product $\langle\cdot, \cdot\rangle$ and norm denoted by $|\cdot|$ ), assume that:
Hypothesis 1.1. $A: D(A) \subset H \rightarrow H$ is a linear self-adjoint operator on $H$ such that $\langle A x, x\rangle \geqslant \delta|x|^{2}, \forall x \in D(A)$, for some $\delta>0$. Moreover, $A^{-1}$ is of trace class. $\left\{e_{j}\right\}$ is an orthonormal basis in $H$ consisting of eigen-functions for $A$, that is, $A e_{j}=\alpha_{j} e_{j}$, $j \in \mathbb{N}$, where $\alpha_{j} \geqslant \delta$.

[^0]In the following $D \varphi: H \rightarrow H$ is the Fréchet-derivative of a function $\varphi: H \rightarrow \mathbb{R}$. By $C_{b}^{1}(H)$ we shall denote the set of all bounded differentiable functions with continuous and bounded derivatives. For $K \subset H$, the space $C_{b}^{1}(K)$ is defined as the space of restrictions of all functions in $C_{b}^{1}(H)$ to the subset $K . \mu$ will denote the Gaussian measure in $H$ with mean 0 and covariance operator $Q:=\frac{1}{2} A^{-1}$. For $\rho \in L_{+}^{1}(H, \mu)$, we consider $\mathcal{E}^{\rho}(u, v)=\frac{1}{2} \int_{H}\langle D u, D v\rangle \rho(z) \mu(\mathrm{d} z), u, v \in C_{b}^{1}(F)$, where $F=\operatorname{Supp}[\rho \cdot \mu]$ and $L_{+}^{1}(H, \mu)$ denotes the set of all non-negative elements in $L^{1}(H, \mu)$. Let $Q R(H)$ be the set of all functions $\rho \in L_{+}^{1}(H, \mu)$ such that $\left(\mathcal{E}^{\rho}, C_{b}^{1}(F)\right)$ is closable on $L^{2}(F ; \rho \cdot \mu)$. Its closure is denoted by $\left(\mathcal{E}^{\rho}, \mathcal{F}^{\rho}\right)$.

Theorem 1.2. Let $\rho \in Q R(H)$. Then $\left(\mathcal{E}^{\rho}, \mathcal{F}^{\rho}\right)$ is a quasi-regular local Dirichlet form on $L^{2}(F ; \rho \cdot \mu)$ in the sense of [6, IV, Definition 3.1].

By virtue of Theorem 1.2 and [6], there exists a diffusion process $M^{\rho}=\left(X_{t}, P_{z}\right)$ on $F$ associated with the Dirichlet form $\left(\mathcal{E}^{\rho}, \mathcal{F}^{\rho}\right) . M^{\rho}$ will be called distorted OU process on $F$. Since constant functions are in $\mathcal{F}^{\rho}$ and $\mathcal{E}^{\rho}(1,1)=0, M^{\rho}$ is recurrent and conservative. Let $A_{1 / 2}(x):=\int_{0}^{x}(\log (1+s))^{1 / 2} \mathrm{~d} s, x \geqslant 0$, and let $\psi$ be its complementary function, namely, $\psi(y):=$ $\int_{0}^{y}\left(A_{1 / 2}^{\prime}\right)^{-1}(t) \mathrm{d} t=\int_{0}^{y}\left(\exp \left(t^{2}\right)-1\right) \mathrm{d} t$. Define $L(\log L)^{1 / 2}:=\left\{f \mid A_{1 / 2}(|f|) \in L^{1}\right\}, \quad L^{\psi}:=\left\{g \mid \psi(c|g|) \in L^{1}\right.$ for some $\left.c>0\right\}$ (cf. [7]). Let $c_{j}, j \in \mathbb{N}$, be a sequence in [1, $\infty$ ). Define $H_{1}:=\left\{x \in H \mid \sum_{j=1}^{\infty}\left\langle x, e_{j}\right\rangle^{2} c_{j}^{2}<\infty\right\}$, equipped with the inner product $\langle x, y\rangle_{H_{1}}:=\sum_{j=1}^{\infty} c_{j}^{2}\left\langle x, e_{j}\right\rangle\left\langle y, e_{j}\right\rangle$. Then clearly $\left(H_{1},\langle,\rangle_{H_{1}}\right)$ is a Hilbert space such that $H_{1} \subset H$ continuously and densely. Identifying $H$ with its dual we obtain the continuous and dense embeddings $H_{1} \subset H\left(\equiv H^{*}\right) \subset H_{1}^{*}$. It follows that $H_{1}\langle z, v\rangle_{H_{1}^{*}}=\langle z, v\rangle_{H}, \forall z \in H_{1}, v \in H$, and that $\left(H_{1}, H, H_{1}^{*}\right)$ is a Gelfand triple. We also introduce a family of $H$-valued function on $H$ by

$$
\left(C_{b}^{1}\right)_{D(A) \cap H_{1}}=\left\{G: G(z)=\sum_{j=1}^{m} g_{j}(z) l^{j}, g_{j} \in C_{b}^{1}(H), l^{j} \in D(A) \cap H_{1}\right\}
$$

Denote by $D^{*}$ the adjoint of $D: C_{b}^{1}(H) \subset L^{2}(H, \mu) \rightarrow L^{2}(H, \mu ; H)$. For $\rho \in L(\log L)^{1 / 2}(H, \mu)$, we put $V(\rho):=$ $\sup _{G \in\left(C_{b}^{1}\right)_{D(A) \cap H_{1}},\|G\|_{H_{1}} \leqslant 1} \int_{H} D^{*} G(z) \rho(z) \mu(\mathrm{d} z)$. A function $\rho$ on $H$ is called a BV function in the Gelfand triple $\left(H_{1}, H, H_{1}^{*}\right)$ (denoted $\rho \in B V\left(H, H_{1}\right)$ in notation), if $\rho \in L(\log L)^{1 / 2}(H, \mu)$ and $V(\rho)$ is finite. When $H_{1}=H=H_{1}^{*}$, this coincides with the definition of BV functions defined in [1] and clearly $B V(H, H) \subset B V\left(H, H_{1}\right)$. This definition is a modification of BV function in abstract Wiener space introduced in [3] and [4].

Theorem 1.3. (i) Suppose $\rho \in B V\left(H, H_{1}\right) \cap L_{+}^{1}(H, \mu)$, then there exist a positive finite measure $\|\mathrm{d} \rho\|$ on $H$ and a Borel-measurable map $\sigma_{\rho}: H \rightarrow H_{1}^{*}$ such that $\left\|\sigma_{\rho}(z)\right\|_{H_{1}^{*}}=1\|\mathrm{~d} \rho\|$-a.e., $V(\rho)=\|\mathrm{d} \rho\|(H)$,

$$
\begin{equation*}
\int_{H} D^{*} G(z) \rho(z) \mu(\mathrm{d} z)=\int_{H} H_{1}\left\langle G(z), \sigma_{\rho}(z)\right\rangle_{H_{1}^{*}}\|\mathrm{~d} \rho\|(\mathrm{d} z), \quad \forall G \in\left(C_{b}^{1}\right)_{D(A) \cap H_{1}} \tag{1.1}
\end{equation*}
$$

Further, if $\rho \in Q R(H),\|\mathrm{d} \rho\|$ is $\mathcal{E}^{\rho}$-smooth, also, $\sigma_{\rho}$ and $\|\mathrm{d} \rho\|$ are uniquely determined.
(ii) Conversely, if Eq. (1.1) holds for $\rho \in L(\log L)^{1 / 2}(H, \mu)$ and for some positive finite measure $\|\mathrm{d} \rho\|$ and a map $\sigma_{\rho}$ with the stated properties, then $\rho \in B V\left(H, H_{1}\right)$ and $V(\rho)=\|\mathrm{d} \rho\|(H)$.

Theorem 1.4. Let $\rho \in Q R(H) \cap B V\left(H, H_{1}\right)$ and consider the measure $\|\mathrm{d} \rho\|$ and $\sigma_{\rho}$ from Theorem 1.3(i). Then there is an $\mathcal{E}^{\rho}$-exceptional set $S \subset F$ such that $\forall z \in F \backslash S$, under $P_{z}$ there exists an $\mathcal{M}_{t}$-cylindrical Wiener process $W^{z}$, such that the sample paths of the associated distorted $O U$-process $M^{\rho}$ on $F$ satisfy the following: for $l \in D(A) \cap H_{1}$

$$
\left\langle l, X_{t}-X_{0}\right\rangle=\int_{0}^{t}\left\langle l, \mathrm{~d} W_{s}^{z}\right\rangle+\frac{1}{2} \int_{0}^{t} H_{1}\left\langle l, \sigma_{\rho}\left(X_{S}\right)\right\rangle_{H_{1}^{*}} \mathrm{~d} L_{s}^{\|\mathrm{d} \rho\|}-\int_{0}^{t}\left\langle A l, X_{S}\right\rangle \mathrm{d} s, \quad \forall t \geqslant 0, P_{z}-a . s .
$$

Here $L_{t}^{\|\mathrm{d} \rho\|}$ is the real valued PCAF associated with $\|\mathrm{d} \rho\|$ by the Revuz correspondence.

## 2. Reflected OU process

Consider the situation when $\rho=I_{\Gamma}$, the indicator of a set.
Remark 2.1. We emphasize that if $\Gamma$ is a convex closed set in $H$, then for each $z, l \in H$ the set $\{s \in \mathbb{R} \mid z+s l \in \Gamma\}$ is a closed interval in $\mathbb{R}$, whose indicator function hence trivially has the Hamza property. Hence, in particular, $I_{\Gamma} \in Q R(H)$.

### 2.1. Reflected OU processes on regular convex set

Denote the corresponding objects $\sigma_{\rho},\left\|\mathrm{d} I_{\Gamma}\right\|$ in Theorem 1.3(i) by $-\mathbf{n}_{\Gamma},\|\partial \Gamma\|$, respectively.
Hypothesis 2.1.1. There exists a convex $C^{\infty}$ function $g: H \rightarrow R$ with $g(0)=0, g^{\prime}(0)=0$, and $D^{2} g$ strictly positively definite, that is, $\left\langle D^{2} g(x) h, h\right\rangle \geqslant \gamma|h|^{2}, \forall h \in H$, where $\gamma>0$, such that

$$
\Gamma=\{x \in H: g(x) \leqslant 1\}, \quad \partial \Gamma=\{x \in H: g(x)=1\} .
$$

Moreover, we also suppose that $D^{2} g$ is bounded on $\Gamma$. Finally, we also suppose that $g$ and all its derivatives grow at infinity at most polynomially.

By using [2, Lemma 2.1], we have (1.1) for $\rho=I_{\Gamma}$ with $H=H_{1}$. By the continuity property of surface measure given in [5], we have the following two theorems.

Theorem 2.1.2. Assume Hypothesis 2.1.1. Then $I_{\Gamma} \in B V(H, H) \cap Q R(H)$.
Theorem 2.1.3. Assume Hypothesis 2.1.1. Then there exists an $\mathcal{E}^{\rho}$-exceptional set $S \subset F$ such that $\forall z \in F \backslash S$, under $P_{z}$ there exists an $\mathcal{M}_{t}$-cylindrical Wiener process $W^{z}$, such that the sample paths of the associated reflected $O U$-process $M^{\rho}$ on $F$ with $\rho=I_{\Gamma}$ satisfy the following: for $l \in D(A)$

$$
\left\langle l, X_{t}-X_{0}\right\rangle=\int_{0}^{t}\left\langle l, \mathrm{~d} W_{s}^{z}\right\rangle-\frac{1}{2} \int_{0}^{t}\left\langle l, \mathbf{n}_{\Gamma}\left(X_{s}\right) \mathrm{d} L_{s}^{\|\partial \Gamma\|}\right\rangle-\int_{0}^{t}\left\langle A l, X_{s}\right\rangle \mathrm{ds}, \quad \forall t \geqslant 0, P_{z} \text {-a.e. }
$$

where $\mathbf{n}_{\Gamma}:=\frac{D g}{|D g|}$ is the exterior normal to $\Gamma$, satisfying $\left\langle\mathbf{n}_{\Gamma}(x), x-y\right\rangle \geqslant 0$, for any $y \in \Gamma, x \in \partial \Gamma$ and $\|\partial \Gamma\|=\mu_{\partial \Gamma}$, where $\mu_{\partial \Gamma}$ is the surface measure induced by $\mu$ (cf. $[2,5]$ ).

Let $\Gamma$ satisfy Hypothesis 2.1.1 and $A$ satisfy Hypothesis 1.1. Consider the following stochastic differential inclusion in the Hilbert space $H$,

$$
\left\{\begin{array}{l}
\mathrm{d} X(t)+\left(A X(t)+N_{\Gamma}(X(t))\right) \mathrm{d} t \ni \mathrm{~d} W(t)  \tag{2.1}\\
X(0)=x
\end{array}\right.
$$

where $W(t)$ is a cylindrical Wiener process in $H$ on a filtered probability space $\left(\Omega, \mathcal{F}, \mathcal{F}_{t}, P\right)$ and $N_{\Gamma}(x)$ is the normal cone to $\Gamma$ at $\chi$.

Definition 2.1.4. A pair of continuous $H \times R$ valued and $\mathcal{F}_{t}$-adapted processes $(X(t), L(t)), t \in[0, T]$, is called a solution of (2.1) if the following conditions hold:
(i) $X(t) \in \Gamma, P$-a.s. for all $t \in[0, T]$,
(ii) $L$ is an increasing process with the property $\int_{0}^{t} I_{\partial \Gamma}\left(X_{s}(\omega)\right) \mathrm{d} L_{s}(\omega)=L_{t}(\omega), t \geqslant 0$, and we have for any $l \in D(A)$, $\left\langle l, X_{t}(\omega)-x\right\rangle=\left\langle l, W_{t}(\omega)-\int_{0}^{t} \mathbf{n}_{\Gamma}\left(X_{s}(\omega)\right) \mathrm{d} L_{s}(\omega)\right\rangle-\left\langle A l, \int_{0}^{t} X_{s}(\omega) \mathrm{d} s\right\rangle$ where $\mathbf{n}_{\Gamma}$ is the exterior normal to $\Gamma$, satisfying $\left\langle\mathbf{n}_{\Gamma}(x), x-y\right\rangle \geqslant 0, \forall y \in \Gamma, x \in \partial \Gamma$.

Theorem 2.1.5. If $\Gamma$ satisfies Hypothesis 2.1.1, then there exists $M, I_{\Gamma} \cdot \mu(M)=1$, such that for every $x \in M$, (2.1) has a pathwise unique continuous strong solution in the sense of Definition 2.1.4. Moreover $X(t) \in M$ for all $t \geqslant 0, P_{\chi}$-a.s.

Remark 2.1.6. We can extend all these results to non-symmetric Dirichlet forms obtained by first order perturbation of the above Dirichlet form.

### 2.2. Reflection OU processes on a class of convex sets

Now we consider the case when $H=L^{2}(0,1), \rho=I_{K_{\alpha}}$, where $K_{\alpha}=\{f \in H \mid f \geqslant-\alpha\}, \alpha \geqslant 0$, and $A=-\frac{1}{2} \frac{\mathrm{~d}^{2}}{\mathrm{~d} r^{2}}$ with Dirichlet boundary condition on [0, 1]. Take $c_{j}=(j \pi)^{\frac{1}{2}+\varepsilon}$ if $\alpha>0, c_{j}=(j \pi)^{\beta}$ if $\alpha=0$, where $\varepsilon \in\left(0, \frac{3}{2}\right]$ and $\beta \in\left(\frac{3}{2}, 2\right]$ respectively. Then $D(A) \subset H_{1}$ continuously for all $\alpha \geqslant 0$. By using [8, (1), (2)], we can prove the following theorem.

Theorem 2.2.1. $I_{K_{\alpha}} \in B V\left(H, H_{1}\right) \cap Q R(H)$.
Remark 2.2.2. It has been proved by Guan Qingyang that $I_{K_{\alpha}}$ is not in $B V(H, H)$. Since we have Theorem 2.2.1, we denote the corresponding objects $\sigma_{\rho},\left\|\mathrm{d} I_{K_{\alpha}}\right\|$ in Theorem 1.3(i) by $n_{\alpha},\left|\sigma_{\alpha}\right|$, respectively.

Theorem 2.2.3. Let $\rho=I_{K_{\alpha}}$. Then there is an $\mathcal{E}^{\rho}$-exceptional set $S \subset F$ such that $\forall z \in F \backslash S$, under $P_{z}$ there exists an $\mathcal{M}_{t}$-cylindrical Wiener process $W^{z}$, such that the sample paths of the associated distorted $O U$-process $M^{\rho}$ on $F$ satisfy the following: for $l \in D(A)$

$$
\left\langle l, X_{t}-X_{0}\right\rangle=\int_{0}^{t}\left\langle l, \mathrm{~d} W_{s}\right\rangle+\frac{1}{2} \int_{0}^{t} H_{1}\left\langle l, n_{\alpha}\left(X_{s}\right)\right\rangle_{H_{1}^{*}} \mathrm{~d} L_{s}^{\left|\sigma_{\alpha}\right|}-\int_{0}^{t}\left\langle A l, X_{s}\right\rangle \mathrm{d} s, \quad P_{z}-a . e .
$$

Here, $L_{t}^{\left|\sigma_{\alpha}\right|}(\omega)$ is a real valued PCAF associated with $\left|\sigma_{\alpha}\right|$ by the Revuz correspondence, satisfying $I_{\left\{X_{s}+\alpha \neq 0\right\}} \mathrm{d} L_{s}^{\left|\sigma_{\alpha}\right|}=0$, and for every $z \in F, P_{z}\left[X_{t} \in C_{0}[0,1]\right.$ for a.e. $\left.t \in[0, \infty)\right]=1$.

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## References

[1] L. Ambrosio, G. Da Prato, D. Pallara, BV functions in a Hilbert space with respect to a Gaussian measure, preprint.
[2] V. Barbu, G. Da Prato, L. Tubaro, Kolmogorov equation associated to the stochastic reflection problem on a smooth convex set of a Hilbert spaces, The Annals of Probability 4 (2009) 1427-1458.
[3] M. Fukushima, BV functions and distorted Ornstein-Uhlenbecl processes over the abstract Wiener space, Journals of Functional Analysis 174 (2000) 227-249.
[4] M. Fukushima, Masanori Hino, On the space of BV functions and a related stochastic calculus in infinite dimensions, Journals of Functional Analysis 183 (2001) 245-268.
[5] P. Malliavin, Stochastic Analysis, Springer, Berlin, 1997.
[6] Z.M. Ma, M. Röckner, Introduction to the Theory of (Non-symmetric) Dirichlet Forms, Springer-Verlag, Berlin/Heidelberg/New York, 1992.
[7] M.M. Rao, Z.D. Ren, Theory of Orlicz Spaces, Monographs and Textbooks in Pure and Applied Mathematics, vol. 146, Dekker, New York, 1991.
[8] L. Zambotti, Integration by parts formulae on convex sets of paths and applications to SPDEs with reflection, Probability Theory Related Fields 123 (2002) 579-600.


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