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Partial Differential Equations/Optimal Control

Observability estimate for stochastic Schrödinger equations

Inégalité d'observabilité pour les équations de Schrödinger stochastiques

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ABSTRACT

In this Note, we present an observability estimate for stochastic Schrödinger equations with nonsmooth lower order terms. The desired inequality is derived by a global Carleman estimate which is based on a fundamental weighted identity for stochastic Schrödinger-like operator. As an interesting byproduct, starting from this identity, one can deduce all the known controllability/observability results for several stochastic and deterministic partial differential equations that are derived before via Carleman estimate in the literature.

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RÉSUMÉ

Dans cette Note, nous établissons une inégalité d'observabilité pour les équations de Schrödinger stochastiques avec des termes d'ordre inférieur à coefficients non réguliers. Notre inégalité s'obtient à partir des inégalités de Carleman globales qui découlent d'une identité à poids pour les opérateurs de type Schrödinger stochastiques. Comme conséquence intéressante de cette identité, on retrouve tous les résultats connus de controlabilité/observabilité pour les équations aux dérivées partielles stochastiques et déterministes, de type Schrödinger et hyperboliques, où l'utilisation des inégalités de Carleman a joué un rôle.

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1. Introduction and the main result

Let T > 0, $G \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a given bounded domain with a C^2 boundary Γ . Put $Q \triangleq (0, T) \times G$ and $\Sigma \triangleq (0, T) \times \Gamma$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge 0}, P)$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{B(t)\}_{t \ge 0}$ is defined. Given a Banach space H, denote by $L^2_{\mathcal{F}}(0, T; H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t \ge 0}$ -adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;H)}) < \infty$, with the canonical norm. Similarly, we denote by $L^\infty_{\mathcal{F}}(0, T; H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t \ge 0}$ -adapted bounded processes, and by $L^2_{\mathcal{F}}(\Omega; C([0, T]; H))$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t \ge 0}$ -adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{C(0,T;H)}) < \infty$.

Consider the following stochastic Schrödinger equation:

$$\begin{cases} i \, dy + \Delta y \, dt = (a_1 \cdot \nabla y + a_2 y + f) \, dt + (a_3 y + g) \, dB(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G, \end{cases}$$
(1)

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with initial datum $y_0 \in H_0^1(G)$, suitable coefficients a_j (j = 1, 2, 3), and source terms f and g. Under suitable conditions (the assumptions in Theorem 1.1 below are enough), one can show that Eq. (1) is well-posed in $H_T \equiv L_{\mathcal{F}}^2(\Omega; C([0, T]; H_0^1(G)))$.

Fix any $x_0 \in \mathbb{R}^d \setminus \overline{G}$. Put $\Gamma_0 = \{x \in \Gamma \mid (x - x_0) \cdot v(x) > 0\}$, where v(x) is the unit outward normal vector of G at $x \in \Gamma$. In this Note, we address to derive a partial boundary observability estimate for Eq. (1), i.e., a quantitative observation from the information on $\Omega \times (0, T) \times \Gamma_0$. Our main result can be stated as follows:

Theorem 1.1. Assume that a_1 is an \mathbb{R}^n -valued function and

$$\begin{cases} a_1 \in L^{\infty}_{\mathcal{F}}(0,T; W^{1,\infty}(G; \mathbb{R}^n)), & a_2 \in L^{\infty}_{\mathcal{F}}(0,T; W^{1,\infty}(G)), & a_3 \in L^{\infty}_{\mathcal{F}}(0,T; W^{1,\infty}(G)), \\ f \in L^2_{\mathcal{F}}(0,T; H^1_0(G)), & g \in L^2_{\mathcal{F}}(0,T; H^1(G)). \end{cases}$$

Then there is a constant $C = C(T, G, \Gamma_0) > 0$ such that for any initial datum $y_0 \in H_0^1(G)$, the corresponding solution to Eq. (1) satisfies

$$\left| y(T) \right|_{L^{2}(\Omega, \mathcal{F}_{T}, P; H_{0}^{1}(G))} \leq \mathcal{C}(a_{1}, a_{2}, a_{3}) \left[\left| \frac{\partial y}{\partial \nu} \right|_{L^{2}_{\mathcal{F}}(0, T; L^{2}(\Gamma_{0}))} + |f|_{L^{2}_{\mathcal{F}}(0, T; H_{0}^{1}(G))} + |g|_{L^{2}_{\mathcal{F}}(0, T; H^{1}(G))} \right],$$

$$(2)$$

where

$$\mathcal{C}(a_1, a_2, a_3) = C \exp \Big[C \Big(|a_1|^2_{L^{\infty}_{\mathcal{F}}(0, T; W^{1,\infty}(G; \mathbb{R}^n))} + |a_2|^2_{L^{\infty}_{\mathcal{F}}(0, T; W^{1,\infty}(G))} + |a_3|^2_{L^{\infty}_{\mathcal{F}}(0, T; W^{1,\infty}(G))} + 1 \Big) \Big].$$

Observability inequality (2) in Theorem 1.1 allows estimating the total energy of solutions at time *T* in terms of the partial energy localized in the observation sub-boundary Γ_0 . This sort of inequality is strongly relevant for control problems. In particular, in the linear setting, this inequality is equivalent to the so-called null controllability property for the backward stochastic Schrödinger equation, i.e., that of driving solutions to rest by means of control forces localized in Γ_0 . This type of inequality, with explicit estimates on the observability constant like that for $C(a_1, a_2, a_3)$ in the above, is also relevant for the control of semilinear problems. Similar inequalities are also useful for solving some Inverse Problems.

In the deterministic case, there exist many approaches and results addressing the observability estimate for Schrödinger equations. For example, results in the spirit of Theorem 1.1 are obtained by Carleman estimate [2,6,10], by the classical Rellich-type multiplier approach [9], by the microlocal analysis approach [7,11], and so on. We refer to [15] for a nice survey in this respect.

One of the main difficulties to establish observability estimates for (ltô-type) stochastic Partial Differential Equations (PDEs for short) is the time-irreversibility of the underlying filtration. Because of this, there exist only a very few works in this direction. To the best of our knowledge, [1,12,14] are the only references addressing the observability estimate of stochastic PDEs, in which [1,12] are devoted to the stochastic parabolic equation; while [14] is for the stochastic wave equation. As far as we know, there is no observability result for stochastic Schrödinger equations in the literature.

We refer to [8] for a detailed proof of Theorem 1.1 and other related results.

2. Carleman estimate for stochastic Schrödinger equations and proof of Theorem 1.1

We begin with the following fundamental weighted identity for a stochastic Schrödinger-like operator, which plays a key role in our proof of Theorem 1.1.

Proposition 2.1. Let $\beta(t, x) \in C^1(\mathbb{R}^{1+m}; \mathbb{R})$, $b^{jk}(t, x) \in C^1(\mathbb{R}^{1+m}; \mathbb{R})$ satisfying $b^{jk} = b^{kj}$ (j, k = 1, 2, ..., m). Assume z is an $H^2_{loc}(\mathbb{R}^m)$ -valued $\{\mathcal{F}_t\}_{t \ge 0}$ -adapted process. Define a formal differential operator \mathcal{P} by $\mathcal{P}z = i\beta(t, x) dz + \sum_{j,k=1}^m (b^{jk}(t, x)z_{x_j})_{x_k} dt$. Let $\ell, \Psi \in C^2(\mathbb{R}^{1+m}; \mathbb{R})$. Put $\theta = e^{\ell}$ and $v = \theta z$. Then for a.e. $x \in \mathbb{R}^m$ and P-a.s. $\omega \in \Omega$, it holds

$$\theta(\mathcal{P}zI_{1} + \mathcal{P}zI_{1}) + dM + \nabla \cdot V$$

$$= 2|I_{1}|^{2} dt + \sum_{j,k=1}^{m} c^{jk} (v_{x_{j}}\overline{v}_{x_{k}} + v_{x_{k}}\overline{v}_{x_{j}}) dt + B|v|^{2} dt + i \sum_{j,k=1}^{m} \left[\left(\beta b^{jk} \ell_{x_{j}}\right)_{t} + b^{jk} (\beta \ell_{t})_{x_{j}} \right] (\overline{v}_{x_{k}}v - v_{x_{k}}\overline{v}) dt$$

$$+ i \left[\beta \Psi + \sum_{j,k=1}^{m} \left(\beta b^{jk} \ell_{x_{j}}\right)_{x_{k}} \right] (\overline{v} dv - v d\overline{v}) + \left(\beta^{2} \ell_{t}\right) dv d\overline{v}, \qquad (3)$$

where $I_1 = -i\beta \ell_t v - 2\sum_{j,k=1}^m b^{jk} \ell_{x_j} v_{x_k} + \Psi v$, $A = \sum_{j,k=1}^m b^{jk} \ell_{x_j} \ell_{x_k} - \sum_{j,k=1}^m (b^{jk} \ell_{x_j})_{x_k} - \Psi$,

$$\begin{cases} M = \beta^{2} \ell_{t} |v|^{2} + i\beta \sum_{j,k=1}^{m} b^{jk} \ell_{x_{j}}(\overline{v}_{x_{k}}v - v_{x_{k}}\overline{v}), & V = [V^{1}, \dots, V^{k}, \dots, V^{m}], \\ V^{k} = -i\beta \sum_{j=1}^{m} [b^{jk} \ell_{x_{j}}(v \, \mathrm{d}\overline{v} - \overline{v} \, \mathrm{d}v) + b^{jk} \ell_{t}(v_{x_{j}}\overline{v} - \overline{v}_{x_{j}}v) \, \mathrm{d}t] - \Psi \sum_{j=1}^{m} b^{jk}(v_{x_{j}}\overline{v} + \overline{v}_{x_{j}}v) \, \mathrm{d}t \\ + \sum_{j=1}^{m} b^{jk} (2A\ell_{x_{j}} + \Psi_{x_{j}}) |v|^{2} \, \mathrm{d}t + \sum_{j,j',k'=1}^{m} (2b^{jk'}b^{j'k} - b^{jk}b^{j'k'}) \ell_{x_{j}}(v_{x_{j'}}\overline{v}_{x_{k'}} + \overline{v}_{x_{j'}}v_{x_{k'}}) \, \mathrm{d}t \end{cases}$$
(4)

and

$$c^{jk} = \sum_{j',k'=1}^{m} \left[2(b^{j'k}\ell_{x_{j'}})_{x_{k'}}b^{jk'} - (b^{jk}b^{j'k'}\ell_{x_{j'}})_{x_{k'}} - b^{jk}\Psi \right],$$

$$B = (\beta^{2}\ell_{t})_{t} + \sum_{j,k=1}^{m} (b^{jk}\Psi_{x_{k}})_{x_{j}} + 2 \left[\sum_{j,k=1}^{m} (b^{jk}\ell_{x_{j}}A)_{x_{k}} + A\Psi \right].$$
(5)

Proof. By $v = \theta z$, it is easy to show that $\theta \mathcal{P} z = I_1 dt + I_2$, where $I_2 = i\beta dv + \sum_{j,k=1}^m (b^{jk}v_{x_j})_{x_k} dt + Av dt$. Hence $\theta(\mathcal{P} z \overline{I_1} + \overline{\mathcal{P} z} I_1) = 2|I_1|^2 dt + (I_1 \overline{I_2} + I_2 \overline{I_1})$, which yields Proposition 2.1 by means of a long but direct computation. \Box

Remark 1. Since we assume only the symmetry condition for $b^{jk}(t, x)$ (without the elliptic assumption), similar to [3], starting from identity (3) in Proposition 2.1, we can deduce, in one shot, controllability/observability results not only for the stochastic Schrödinger equation, but also for deterministic hyperbolic, Schrödinger and plate equations that are derived before via Carleman estimate in the literature, i.e., that appeared in [4,6] and [13], respectively.

We now present a global Carleman estimate for stochastic Schrödinger equations. Because of the time-irreversibility of Eq. (1), following [5,12,14], we choose a weight function vanishing at time t = 0. Put $\psi(x) = |x - x_0|^2 + \gamma$, where γ is a positive number such that $\psi > \frac{2}{3} |\psi|_{L^{\infty}(G)} > 0$ in *G*. Also, choose

$$\ell = s \frac{e^{4\lambda\psi} - e^{5\lambda|\psi|_{L^{\infty}(G)}}}{t^2(T-t)^2}, \qquad \varphi = \frac{e^{4\lambda\psi}}{t^2(T-t)^2}.$$
(6)

Choose $\beta = 1$, m = n and $(b^{jk})_{1 \le j, k \le m} = I_n$, the $n \times n$ identity matrix. Applying Proposition 2.1 to Eq. (1) with z replaced by y and $v = \theta z$. Integrating identity (3) in Q, taking the mean value and choosing s and λ large enough, we arrive at the desired global Carleman estimate for Eq. (1):

$$\mathbb{E} \int_{Q} \theta^{2} \left(s\lambda^{2} \varphi |\nabla y|^{2} + s^{3} \lambda^{4} \varphi^{3} |y|^{3} \right) dx dt$$

$$\leq C \left\{ \mathbb{E} \int_{Q} \theta^{2} |f|^{2} dx dt + \mathbb{E} \int_{Q} \theta^{2} \left[s^{2} \lambda^{2} \varphi^{2} g^{2} + |\nabla g|^{2} \right] dx dt + \mathbb{E} \int_{0}^{T} \int_{\Gamma_{0}} \theta^{2} s\lambda \varphi \left| \frac{\partial y}{\partial \nu} \right|^{2} d\Gamma_{0} dt \right\}.$$
(7)

Finally, by (7) and using the usual energy method, we conclude Theorem 1.1.

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