# Functional equations for zeta functions of $\mathbb{F}_{1}$-schemes 

## Équations fonctionnelles pour les fonctions zêta de schémas sur $\mathbb{F}_{1}$

## Oliver Lorscheid

The City College of New York, Department of Mathematics, 160 Convent Avenue, New York, NY 10031, USA

## A R T I C L E IN F O

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#### Abstract

For a scheme $X$ whose $\mathbb{F}_{q}$-rational points are counted by a polynomial $N(q)=\sum a_{i} q^{i}$, the $\mathbb{F}_{1}$-zeta function is defined as $\zeta \mathcal{X}(s)=\Pi(s-i)^{-a_{i}}$. Define $\chi=N(1)$. In this paper we show that if $X$ is a smooth projective scheme, then its $\mathbb{F}_{1}$-zeta function satisfies the functional equation $\zeta_{\mathcal{X}}(n-s)=(-1)^{\chi} \zeta_{\mathcal{X}}(s)$. We further show that the $\mathbb{F}_{1}$-zeta function $\zeta_{\mathcal{G}}(s)$ of a split reductive group scheme $G$ of rank $r$ with $N$ positive roots satisfies the functional equation $\zeta_{\mathcal{G}}(r+N-s)=(-1)^{\chi}\left(\zeta_{\mathcal{G}}(s)\right)^{(-1)^{r}}$.


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## RÉS U M É

Pour un schéma $X$ dont les points $\mathbb{F}_{q}$-rationnels sont comptés par un polynôme $N(q)=$ $\sum a_{i} q^{i}$, la fonction zêta sur $\mathbb{F}_{1}$ est définie par $\zeta \mathcal{X}(s)=\Pi(s-i)^{-a_{i}}$. Posons $\chi=N(1)$. Dans cette Note nous montrons que si $X$ est un schéma projectif lisse, alors sa fonction zêta sur $\mathbb{F}_{1}$ satisfait l'équation fonctionnelle $\zeta \mathcal{X}(n-s)=(-1)^{\chi} \zeta \mathcal{X}(s)$. Nous montrons aussi que la fonction zêta $\zeta_{\mathcal{G}}(s)$ sur $\mathbb{F}_{1}$ d'un schéma en groupes réductif déployé $G$ de rang $r$ avec $N$ racines positives satisfait l'équation fonctionnelle $\zeta_{\mathcal{G}}(r+N-s)=(-1)^{\chi}\left(\zeta_{\mathcal{G}}(s)\right)^{(-1)^{r}}$.
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## 1. Introduction

In recent years around a dozen different suggestions of what a scheme over $\mathbb{F}_{1}$ should be appeared in literature (cf. [6]). The common motivation for all these approaches is to provide a framework in which Deligne's proof of the Weyl conjectures can be transfered to characteristic 0 in order to proof the Riemann hypothesis. Roughly speaking, $\mathbb{F}_{1}$ should be thought of as a field of coefficients for $\mathbb{Z}$, and $\mathbb{F}_{1}$-schemes $\mathcal{X}$ should have a base extension $\mathcal{X}_{\mathbb{Z}}$ to $\mathbb{Z}$ which is a scheme in the usual sense.

Though it is not clear yet whether any of the existing $\mathbb{F}_{1}$-geometries comes close towards realizing a proof of the Riemann hypothesis, and thus, in particular, it is not clear what the appropriate notion of an $\mathbb{F}_{1}$-scheme should be, the zeta function $\zeta \mathcal{X}(s)$ of such an elusive $\mathbb{F}_{1}$-scheme $\mathcal{X}$ is determined by the scheme $X=\mathcal{X}_{\mathbb{Z}}$.

Namely, let $X$ be a variety of dimension $n$ over $\mathbb{Z}$, i.e. a scheme such that $X_{k}$ is a variety of dimension $n$ for any field $k$. Assume further that $X$ has a counting polynomial

[^0]$$
N(q)=\sum_{i=0}^{n} a_{i} q^{i} \in \mathbb{Z}[q]
$$
i.e. the number of $\mathbb{F}_{q}$-rational points is counted by $\# X\left(\mathbb{F}_{q}\right)=N(q)$ for every prime power $q$. If $X$ descends to an $\mathbb{F}_{1}$-scheme $\mathcal{X}$, i.e. $\mathcal{X}_{\mathbb{Z}} \simeq X$, then $\mathcal{X}$ has the zeta function
$$
\zeta_{\mathcal{X}}(s)=\lim _{q \rightarrow 1}(q-1)^{\chi} \zeta_{X}(q, s)
$$
where $\zeta_{X}(q, s)=\exp \left(\sum_{r \geqslant 1} N\left(q^{r}\right) q^{-s r} / r\right)$ is the zeta function of $X \otimes \mathbb{F}_{q}$ if $q$ is a prime power and $\chi=N(1)$ is the order the pole of $\zeta_{X}(q, s)$ in $q=1$ (cf. [9]). This expression comes down to
$$
\zeta \mathcal{X}(s)=\prod_{i=0}^{n}(s-i)^{-a_{i}}
$$
[9, Lemme 1].
From this it is clear that $\zeta \mathcal{X}(s)$ is a rational function in $s$ and that its zeros (resp. poles) are at $s=i$ of order $-a_{i}$ for $i=0, \ldots, n$. The only statement from the Weyl conjectures which is not obvious for zeta functions of $\mathbb{F}_{1}$-schemes is the functional equation.

## 2. The functional equation for smooth projective $\mathbb{F}_{1}$-schemes

Let $X$ be an (irreducible) smooth projective variety of dimesion $n$ with a counting polynomial $N(q)$. Let $b_{0}, \ldots, b_{2 n}$ be the Betti numbers of $X$, i.e. the dimensions of the singular homology groups $H_{0}(X(\mathbb{C})), \ldots, H_{2 n}(X(\mathbb{C}))$. By Poincaré duality, we know that $b_{2 n-i}=b_{i}$. As a consequence of the comparison theorem for smooth liftable varieties and Deligne's proof of the Weil conjectures, we know that the counting polynomial is of the form

$$
N(q)=\sum_{i=0}^{n} b_{2 i} q^{i}
$$

and that $b_{i}=0$ if $i$ is odd (cf. [2] and [8]). Thus $\chi=\sum_{i=0}^{n} b_{2 i}$ is the Euler characteristic of $X_{\mathbb{C}}$ in this case (cf. [4]).
Suppose $X$ has an elusive model $\mathcal{X}$ over $\mathbb{F}_{1}$. Then $\mathcal{X}$ has the zeta function $\zeta \mathcal{X}(s)=\prod_{i=0}^{n}(s-i)^{-b_{2 i}}$.
Theorem 1. The zeta function $\zeta \mathcal{X}(s)$ satisfies the functional equation

$$
\zeta \mathcal{X}(n-s)=(-1)^{\chi} \zeta \mathcal{X}(s)
$$

and the factor equals -1 if and only if $n$ is even and $b_{n}$ is odd.
Proof. We calculate

$$
\zeta_{\mathcal{X}}(n-s)=\prod_{i=0}^{n}((n-s)-i)^{-b_{2 i}}=\prod_{i=0}^{n}(-1)^{b_{2 i}}(s-(n-i))^{-b_{2 i}}=(-1)^{\chi} \prod_{i=0}^{d}(s-(n-i))^{-b_{2 n-2 i}}
$$

where we used $b_{2 n-2 i}=b_{2 i}$ in the last equation. If we substitute $i$ by $n-i$ in this expression, we obtain

$$
\zeta \mathcal{X}(n-s)=(-1)^{\chi} \prod_{i=0}^{n}(s-i)^{-b_{2 i}}=(-1)^{\chi} \zeta_{\mathcal{X}}(s)
$$

If $n$ is odd, then there is an even number of non-trivial Betti numbers and $\chi=2 b_{0}+2 b_{2}+\cdots+2 b_{n-1}$ is even. If $n$ is odd, then $\chi=2 b_{0}+2 b_{2}+\cdots+2 b_{n-2}+b_{n}$ has the same parity as $b_{n}$. Thus the additional statement.

Remark 2. Note the similarity with the functional equation for motivic zeta functions as in [3, Theorem 1]. Amongst other factors, $(-1)^{\chi(M)}$ appears also in the functional equation of the zeta function of a motive $M$ where $\chi(M)$ is the (positive part of the) Euler characteristic of $M$.

## 3. The functional equation for reductive groups over $\mathbb{F}_{1}$

The above observations imply further a functional equation for reductive group schemes over $\mathbb{F}_{1}$. Note that Soulé's and Connes and Consani's approaches towards $\mathbb{F}_{1}$-geometry indeed succeeded in descending split reductive group schemes from $\mathbb{Z}$ to $\mathbb{F}_{1}$ (cf. [1,5,7]).

Let $G$ be a split reductive group scheme of rank $r$ with Borel group $B$ and maximal split torus $T \subset B$. Let $N$ be the normalizer of $T$ in $G$ and $W=N(\mathbb{Z}) / T(\mathbb{Z})$ be the Weyl group. The Bruhat decomposition of $G$ (with respect to $T$ and $B$ ) is the morphism

$$
\coprod_{w \in W} B w B \rightarrow G
$$

induced by the subscheme inclusions $B w B \rightarrow G$, which has the property that it induces a bijection between the $k$-rational points for every field $k$. We have $B \simeq \mathbb{G}_{m}^{r} \times \mathbb{A}^{N}$ as schemes where $N$ is the number of positive roots of $G$, and $B w B \simeq$ $\mathbb{G}_{m}^{r} \times \mathbb{A}^{N+\lambda(w)}$ where $\lambda(w)$ is the length of $w \in W$. With this we can calculate the counting polynomial of $G$ as

$$
N(q)=\# \coprod_{w \in W} B w B\left(\mathbb{F}_{q}\right)=(q-1)^{r} q^{N} \sum_{w \in W} q^{\lambda(w)}
$$

The quotient variety $G / B$ is a smooth projective scheme of dimension $N$ with counting function $N_{G / B}(q)=$ $\left((q-1)^{r} q^{N}\right)^{-1} N(q)=\sum_{w \in W} q^{\lambda(w)}$. Let $b_{0}, \ldots, b_{2 N}$ be the Betti numbers of $G / B$, then we know from the previous section that $N_{G / B}(q)=\sum_{l=0}^{N} b_{2 l} q^{l}$ and that $b_{2 N-2 l}=b_{2 l}$.

Thus we obtain for the counting polynomial of $G$ that

$$
N(q)=q^{N}\left(\sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k} q^{k}\right) \cdot\left(\sum_{l=0}^{N} b_{2 l} q^{l}\right)=\sum_{i=0}^{d}\left(\sum_{k+l=i-N}(-1)^{r-k}\binom{r}{k} b_{2 l}\right) q^{i}
$$

where $d=r+2 N$ is the dimension of $G$ and with the convention that $\binom{r}{k}=0$ if $k<0$ or $k>r$. Denote by $a_{i}=$ $\sum_{k+l=i-N}(-1)^{r-k}\binom{r}{k} b_{2 l}$ the coefficients of $N(q)$.

Lemma 3.1. We have $a_{0}=\cdots=a_{N-1}=0$ and $a_{d-i}=(-1)^{r} a_{i+N}$.
Proof. The first statement follows from the fact that $N(q)$ is divisible by $q^{N}$. For the second statement we use the symmetries $\binom{r}{k}=\binom{r}{r-k}$ and $b_{2 N-2 l}=b_{2 l}$ to calculate

$$
a_{d-i}=\sum_{k+l=d-i-N}(-1)^{r-k}\binom{r}{k} b_{2 l}=\sum_{k+l=d-i-N}(-1)^{r}(-1)^{k}\binom{r}{r-k} b_{2 N-2 l} .
$$

When we substitute $k$ by $r-k$ and $l$ by $N-l$ in this equation and use $d=r+2 N$, we obtain

$$
a_{d-i}=(-1)^{r} \sum_{k+l=(i+N)-N}(-1)^{r-k}\binom{r}{k} b_{2 l}
$$

which is the same as $(-1)^{r} a_{i+N}$.
Suppose $G$ has an elusive model $\mathcal{G}$ over $\mathbb{F}_{1}$. Then $\mathcal{G}$ has the zeta function $\zeta_{\mathcal{G}}(s)=\prod_{i=0}^{n}(s-i)^{-a_{i}}$. Let $\chi=N(1)=\sum_{i=0}^{d} a_{i}$.
Theorem 3. The zeta function $\zeta_{\mathcal{G}}(s)$ satisfies the functional equation

$$
\zeta_{\mathcal{G}}(r+N-s)=(-1)^{\chi}\left(\zeta_{\mathcal{G}}(s)\right)^{(-1)^{r}}
$$

Proof. We use the previous lemma and $r+N=d-N$ to calculate that

$$
\zeta_{\mathcal{G}}(r+N-s)=\prod_{i=0}^{n}(r+N-s-i)^{-a_{i}}=\prod_{i=0}^{n}(d-N-s-i)^{-(-1)^{r} a_{d-N-i}} .
$$

After substituting $i$ by $d-N-i$, we find that

$$
\zeta_{\mathcal{G}}(r+N-s)=\prod_{i=0}^{n}(i-s)^{-(-1)^{r} a_{i}}=(-1)^{\sum a_{i}}\left(\prod_{i=0}^{n}(s-i)^{-a_{i}}\right)^{(-1)^{r}}=(-1)^{\chi}\left(\zeta_{\mathcal{G}}(s)\right)^{(-1)^{r}} .
$$

Remark 4. Kurokawa calculates the $\mathbb{F}_{1}$-zeta functions of $\mathbb{P}^{n}, \mathrm{GL}(n)$ and $\operatorname{SL}(n)$ in [4]. One can verify the functional equation for these examples immediately.

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[^0]:    E-mail address: olorscheid@ccny.cuny.edu.
    URL: http://wmaz.math.uni-wuppertal.de/lorscheid/ccny/.

