Functional equations for zeta functions of $\mathbb{F}_1$-schemes

Équations fonctionnelles pour les fonctions zêta de schémas sur $\mathbb{F}_1$

Oliver Lorscheid

The City College of New York, Department of Mathematics, 160 Convent Avenue, New York, NY 10031, USA

ABSTRACT

For a scheme $X$ whose $\mathbb{F}_q$-rational points are counted by a polynomial $N(q) = \sum a_i q^i$, the $\mathbb{F}_1$-zeta function is defined as $\zeta_X(s) = \prod (s - i)^{-a_i}$. Define $\chi = N(1)$. In this paper we show that if $X$ is a smooth projective scheme, then its $\mathbb{F}_1$-zeta function satisfies the functional equation $\zeta_X(n - s) = (-1)^s \chi \zeta_X(s)$. We further show that the $\mathbb{F}_1$-zeta function $\zeta_G(s)$ of a split reductive group scheme $G$ of rank $r$ with $N$ positive roots satisfies the functional equation $\zeta_G(r + N - s) = (-1)^s (\zeta_G(s))^{1/r}.\$ © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

RÉSUMÉ

Pour un schéma $X$ dont les points $\mathbb{F}_q$-rationnels sont comptés par un polynôme $N(q) = \sum a_i q^i$, la fonction zêta sur $\mathbb{F}_1$ est définie par $\zeta_X(s) = \prod (s - i)^{-a_i}$. Posons $\chi = N(1)$. Dans cette Note nous montrons que si $X$ est un schéma projectif lisse, alors sa fonction zêta sur $\mathbb{F}_1$ satisfait l’équation fonctionnelle $\zeta_X(n - s) = (-1)^s \chi \zeta_X(s)$. Nous montrons aussi que la fonction zêta $\zeta_G(s)$ sur $\mathbb{F}_1$ d’un schéma en groupes réductif déployé $G$ de rang $r$ avec $N$ racines positives satisfait l’équation fonctionnelle $\zeta_G(r + N - s) = (-1)^s (\zeta_G(s))^{1/r}.\$ © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

In recent years around a dozen different suggestions of what a scheme over $\mathbb{F}_1$ should be appeared in literature (cf. [6]). The common motivation for all these approaches is to provide a framework in which Deligne’s proof of the Weyl conjectures can be transferred to characteristic 0 in order to proof the Riemann hypothesis. Roughly speaking, $\mathbb{F}_1$ should be thought of as a field of coefficients for $\mathbb{Z}$, and $\mathbb{F}_1$-schemes $\mathcal{X}$ should have a base extension $\mathcal{X}_\mathbb{Z}$ to $\mathbb{Z}$ which is a scheme in the usual sense.

Though it is not clear yet whether any of the existing $\mathbb{F}_1$-geometries comes close towards realizing a proof of the Riemann hypothesis, and thus, in particular, it is not clear what the appropriate notion of an $\mathbb{F}_1$-scheme should be, the zeta function $\zeta_{\mathcal{X}}(s)$ of such an elusive $\mathbb{F}_1$-scheme $\mathcal{X}$ is determined by the scheme $X = \mathcal{X}_\mathbb{Z}$.

Namely, let $X$ be a variety of dimension $n$ over $\mathbb{Z}$, i.e. a scheme such that $X_k$ is a variety of dimension $n$ for any field $k$. Assume further that $X$ has a counting polynomial

E-mail address: olorescheid@ccny.cuny.edu.
URL: http://wmaz.math.uni-wuppertal.de/lorscheid/ccny/.

1631-073X/$ – see front matter © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.crma.2010.10.010
\[ N(q) = \sum_{i=0}^{n} a_i q^i \in \mathbb{Z}[q], \]

i.e. the number of \( \mathbb{F}_q \)-rational points is counted by \( \#X(\mathbb{F}_q) = N(q) \) for every prime power \( q \). If \( X \) descends to an \( \mathbb{F}_1 \)-scheme \( X' \), i.e. \( X' \cong X \), then \( X' \) has the zeta function

\[ \zeta_{X'}(s) = \lim_{q \to 1} (q - 1)^X \zeta_X(q, s) \]

where \( \zeta_X(q, s) = \exp(\sum_{i \geq 1} N(q^i)q^{-sr}) \) is the zeta function of \( X \otimes \mathbb{F}_q \) if \( q \) is a prime power and \( \chi = N(1) \) is the order of the pole of \( \zeta_X(q, s) \) in \( q = 1 \) (cf. [9]). This expression comes down to

\[ \zeta_{X'}(s) = \prod_{i=0}^{n} (s - i)^{-a_i} \]

[9, Lemme 1].

From this it is clear that \( \zeta_{X'}(s) \) is a rational function in \( s \) and that its zeros (resp. poles) are at \( s = i \) of order \( -a_i \) for \( i = 0, \ldots, n \). The only statement from the Weyl conjectures which is not obvious for zeta functions of \( \mathbb{F}_1 \)-schemes is the functional equation.

2. The functional equation for smooth projective \( \mathbb{F}_1 \)-schemes

Let \( X \) be an (irreducible) smooth projective variety of dimension \( n \) with a counting polynomial \( N(q) \). Let \( b_0, \ldots, b_{2n} \) be the Betti numbers of \( X \), i.e. the dimensions of the singular homology groups \( H_0(X(\mathbb{C})), \ldots, H_{2n}(X(\mathbb{C})) \). By Poincaré duality, we know that \( b_{2n-i} = b_i \). As a consequence of the comparison theorem for smooth liftable varieties and Deligne’s proof of the Weil conjectures, we know that the counting polynomial is of the form

\[ N(q) = \sum_{i=0}^{n} b_{2i} q^i \]

and that \( b_1 = 0 \) if \( i \) is odd (cf. [2] and [8]). Thus \( \chi = \sum_{i=0}^{n} b_{2i} \) is the Euler characteristic of \( X_{\mathbb{C}} \) in this case (cf. [4]).

Suppose \( X \) has an elusive model \( X' \) over \( \mathbb{F}_1 \). Then \( X' \) has the zeta function \( \zeta_{X'}(s) = \prod_{i=0}^{n} (s - i)^{-b_{2i}} \).

**Theorem 1.** The zeta function \( \zeta_{X'}(s) \) satisfies the functional equation

\[ \zeta_{X'}(n - s) = (-1)^n \zeta_{X'}(s) \]

and the factor equals \(-1\) if and only if \( n \) is even and \( b_n \) is odd.

**Proof.** We calculate

\[ \zeta_{X'}(n - s) = \prod_{i=0}^{n} ((n - s) - i)^{-b_{2i}} = \prod_{i=0}^{n} (-1)^{b_2} (s - (n - i))^{-b_{2i}} = (-1)^X \prod_{i=0}^{d} (s - (n - i))^{-b_{2n-2i}} \]

where we used \( b_{2n-2i} = b_{2i} \) in the last equation. If we substitute \( i \) by \( n - i \) in this expression, we obtain

\[ \zeta_{X'}(n - s) = (-1)^X \prod_{i=0}^{n} (s - i)^{-b_{2i}} = (-1)^X \zeta_{X'}(s). \]

If \( n \) is odd, then there is an even number of non-trivial Betti numbers and \( \chi = 2b_0 + 2b_2 + \cdots + 2b_{n-1} \) is even. If \( n \) is odd, then \( \chi = 2b_0 + 2b_2 + \cdots + 2b_{n-2} + b_n \) has the same parity as \( b_n \). Thus the additional statement. \( \square \)

**Remark 2.** Note the similarity with the functional equation for motivic zeta functions as in [3, Theorem 1]. Amongst other factors, \((-1)^X(M)\) appears also in the functional equation of the zeta function of a motive \( M \) where \( \chi(M) \) is the (positive part of the) Euler characteristic of \( M \).

3. The functional equation for reductive groups over \( \mathbb{F}_1 \)

The above observations imply further a functional equation for reductive group schemes over \( \mathbb{F}_1 \). Note that Soulé’s and Connes and Consani’s approaches towards \( \mathbb{F}_1 \)-geometry indeed succeeded in descending split reductive group schemes from \( \mathbb{Z} \) to \( \mathbb{F}_1 \) (cf. [1,5,7]).
Let $G$ be a split reductive group scheme of rank $r$ with Borel group $B$ and maximal split torus $T \subset B$. Let $N$ be the normalizer of $T$ in $G$ and $W = N(\mathbb{Z})/T(\mathbb{Z})$ be the Weyl group. The Bruhat decomposition of $G$ (with respect to $T$ and $B$) is the morphism

$$\prod_{w \in W} BwB \to G,$$

induced by the subscheme inclusions $BwB \to G$, which has the property that it induces a bijection between the $k$-rational points for every field $k$. We have $B \simeq G_m^r \times \mathbb{A}^N$ as schemes where $N$ is the number of positive roots of $G$, and $BwB \simeq G_m^r \times \mathbb{A}^{N+\lambda(w)}$ where $\lambda(w)$ is the length of $w \in W$. With this we can calculate the counting polynomial of $G$ as

$$N(q) = \# \prod_{w \in W} BwB(\mathbb{F}_q) = (q - 1)^r q^N \sum_{w \in W} q^{\lambda(w)}.$$

The quotient variety $G/B$ is a smooth projective scheme of dimension $N$ with counting function $N_{G/B}(q) = ((q - 1)^r q^N)^{-1} N(q) = \sum_{w \in W} q^{\lambda(w)}$. Let $b_0, \ldots, b_{2N}$ be the Betti numbers of $G/B$, then we know from the previous section that $N_{G/B}(q) = \sum_{i=0}^N b_i q^i$ and that $b_{2N-2l} = b_{2l}$.

Thus we obtain for the counting polynomial of $G$ that

$$N(q) = q^N \left( \sum_{i=0}^r (-1)^i \binom{r}{i} q^i \right) \cdot \left( \sum_{i=0}^d \sum_{k+i-l-N} (-1)^{r-k} \binom{r}{k} b_{2l} q^i \right)$$

where $d = r + 2N$ is the dimension of $G$ and with the convention that $\binom{r}{k} = 0$ if $k < 0$ or $k > r$. Denote by $a_i = \sum_{k+i-l-N} (-1)^{r-k} \binom{r}{k} b_{2l}$ the coefficients of $N(q)$.

**Lemma 3.1.** We have $a_0 = \cdots = a_{N-1} = 0$ and $a_{d-l} = (-1)^l a_{l+N}$.

**Proof.** The first statement follows from the fact that $N(q)$ is divisible by $q^N$. For the second statement we use the symmetries $\binom{r}{k} = \binom{r}{r-k}$ and $b_{2N-2l} = b_{2l}$ to calculate

$$a_{d-l} = \sum_{k=l-N} (-1)^{r-k} \binom{r}{k} b_{2l} = \sum_{k+l-i-N} (-1)^l (-1)^{r-k} \binom{r}{r-k} b_{2N-2l}.$$

When we substitute $k$ by $r-k$ and $l$ by $N-l$ in this equation and use $d = r + 2N$, we obtain

$$a_{d-l} = (-1)^l \sum_{k+l=(i+N)-N} (-1)^{r-k} \binom{r}{k} b_{2l},$$

which is the same as $(-1)^l a_{l+N}$. □

Suppose $G$ has an elusive model $G$ over $\mathbb{F}_1$. Then $G$ has the zeta function $\zeta_G(s) = \prod_{i=0}^N(s-i)^{-a_i}$. Let $\chi = N(1) = \sum_{i=0}^d a_i$.

**Theorem 3.** The zeta function $\zeta_G(s)$ satisfies the functional equation

$$\zeta_G(r + N - s) = (-1)^r \left( \zeta_G(s) \right)^{(-1)^r}.$$

**Proof.** We use the previous lemma and $r + N = d - N$ to calculate that

$$\zeta_G(r + N - s) = \prod_{i=0}^n (r + N - s - i)^{-a_i} = \prod_{i=0}^n (d - N - s - i)^{-(1)^l a_{d-N-1-i}}.$$

After substituting $i$ by $d - N - i$, we find that

$$\zeta_G(r + N - s) = \prod_{i=0}^n (i - s)^{-(1)^{r-a_i}} = (-1)^{\sum_{i=0}^n a_i} \left( \prod_{i=0}^n (s-i)^{-a_i} \right)^{(-1)^r} = (-1)^\chi \left( \zeta_G(s) \right)^{(-1)^r}. \quad \Box$$

**Remark 4.** Kurokawa calculates the $\mathbb{F}_1$-zeta functions of $\mathbb{P}^n$, $\text{GL}(n)$ and $\text{SL}(n)$ in [4]. One can verify the functional equation for these examples immediately.
Acknowledgements

I like to thank Takashi Ono for drawing my attention to the symmetries occurring in the counting polynomials of split reductive group schemes. I like to thank Markus Reineke for his explanations on the comparison theorem for liftable smooth varieties.

References