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Functional equations for zeta functions of \mathbb{F}_1 -schemes

Équations fonctionnelles pour les fonctions zêta de schémas sur \mathbb{F}_1

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ABSTRACT

For a scheme *X* whose \mathbb{F}_q -rational points are counted by a polynomial $N(q) = \sum a_i q^i$, the \mathbb{F}_1 -zeta function is defined as $\zeta_{\mathcal{X}}(s) = \prod (s-i)^{-a_i}$. Define $\chi = N(1)$. In this paper we show that if *X* is a smooth projective scheme, then its \mathbb{F}_1 -zeta function satisfies the functional equation $\zeta_{\mathcal{X}}(n-s) = (-1)^{\chi} \zeta_{\mathcal{X}}(s)$. We further show that the \mathbb{F}_1 -zeta function $\zeta_{\mathcal{G}}(s)$ of a split reductive group scheme *G* of rank *r* with *N* positive roots satisfies the functional equation $\zeta_{\mathcal{G}}(r+N-s) = (-1)^{\chi} (\zeta_{\mathcal{G}}(s))^{(-1)^r}$.

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RÉSUMÉ

Pour un schéma X dont les points \mathbb{F}_q -rationnels sont comptés par un polynôme $N(q) = \sum a_i q^i$, la fonction zêta sur \mathbb{F}_1 est définie par $\zeta_{\mathcal{X}}(s) = \prod (s-i)^{-a_i}$. Posons $\chi = N(1)$. Dans cette Note nous montrons que si X est un schéma projectif lisse, alors sa fonction zêta sur \mathbb{F}_1 satisfait l'équation fonctionnelle $\zeta_{\mathcal{X}}(n-s) = (-1)^{\chi} \zeta_{\mathcal{X}}(s)$. Nous montrons aussi que la fonction zêta $\zeta_{\mathcal{G}}(s)$ sur \mathbb{F}_1 d'un schéma en groupes réductif déployé *G* de rang *r* avec *N* racines positives satisfait l'équation fonctionnelle $\zeta_{\mathcal{G}}(r+N-s) = (-1)^{\chi} (\zeta_{\mathcal{G}}(s))^{(-1)^r}$. © 2010 Académie des sciences, Published by Elsevier Masson SAS, All rights reserved.

1. Introduction

In recent years around a dozen different suggestions of what a scheme over \mathbb{F}_1 should be appeared in literature (cf. [6]). The common motivation for all these approaches is to provide a framework in which Deligne's proof of the Weyl conjectures can be transfered to characteristic 0 in order to proof the Riemann hypothesis. Roughly speaking, \mathbb{F}_1 should be thought of as a field of coefficients for \mathbb{Z} , and \mathbb{F}_1 -schemes \mathcal{X} should have a base extension $\mathcal{X}_{\mathbb{Z}}$ to \mathbb{Z} which is a scheme in the usual sense.

Though it is not clear yet whether any of the existing \mathbb{F}_1 -geometries comes close towards realizing a proof of the Riemann hypothesis, and thus, in particular, it is not clear what the appropriate notion of an \mathbb{F}_1 -scheme should be, the zeta function $\zeta_{\mathcal{X}}(s)$ of such an elusive \mathbb{F}_1 -scheme \mathcal{X} is determined by the scheme $X = \mathcal{X}_{\mathbb{Z}}$.

Namely, let *X* be a variety of dimension *n* over \mathbb{Z} , i.e. a scheme such that X_k is a variety of dimension *n* for any field *k*. Assume further that *X* has a counting polynomial

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$$N(q) = \sum_{i=0}^{n} a_i q^i \in \mathbb{Z}[q],$$

i.e. the number of \mathbb{F}_q -rational points is counted by $\#X(\mathbb{F}_q) = N(q)$ for every prime power q. If X descends to an \mathbb{F}_1 -scheme \mathcal{X} , i.e. $\mathcal{X}_{\mathbb{Z}} \simeq X$, then \mathcal{X} has the zeta function

$$\zeta_{\mathcal{X}}(s) = \lim_{q \to 1} (q-1)^{\chi} \zeta_X(q,s)$$

where $\zeta_X(q, s) = \exp(\sum_{r \ge 1} N(q^r)q^{-sr}/r)$ is the zeta function of $X \otimes \mathbb{F}_q$ if q is a prime power and $\chi = N(1)$ is the order the pole of $\zeta_X(q, s)$ in q = 1 (cf. [9]). This expression comes down to

$$\zeta_{\mathcal{X}}(s) = \prod_{i=0}^{n} (s-i)^{-a_i}$$

[9, Lemme 1].

From this it is clear that $\zeta_{\mathcal{X}}(s)$ is a rational function in *s* and that its zeros (resp. poles) are at s = i of order $-a_i$ for i = 0, ..., n. The only statement from the Weyl conjectures which is not obvious for zeta functions of \mathbb{F}_1 -schemes is the functional equation.

2. The functional equation for smooth projective \mathbb{F}_1 -schemes

Let *X* be an (irreducible) smooth projective variety of dimesion *n* with a counting polynomial N(q). Let b_0, \ldots, b_{2n} be the Betti numbers of *X*, i.e. the dimensions of the singular homology groups $H_0(X(\mathbb{C})), \ldots, H_{2n}(X(\mathbb{C}))$. By Poincaré duality, we know that $b_{2n-i} = b_i$. As a consequence of the comparison theorem for smooth liftable varieties and Deligne's proof of the Weil conjectures, we know that the counting polynomial is of the form

$$N(q) = \sum_{i=0}^{n} b_{2i} q^i$$

and that $b_i = 0$ if *i* is odd (cf. [2] and [8]). Thus $\chi = \sum_{i=0}^n b_{2i}$ is the Euler characteristic of $X_{\mathbb{C}}$ in this case (cf. [4]). Suppose *X* has an elusive model \mathcal{X} over \mathbb{F}_1 . Then \mathcal{X} has the zeta function $\zeta_{\mathcal{X}}(s) = \prod_{i=0}^n (s-i)^{-b_{2i}}$.

Theorem 1. The zeta function $\zeta_{\mathcal{X}}(s)$ satisfies the functional equation

$$\zeta_{\mathcal{X}}(n-s) = (-1)^{\chi} \zeta_{\mathcal{X}}(s)$$

and the factor equals -1 if and only if n is even and b_n is odd.

Proof. We calculate

$$\zeta_{\mathcal{X}}(n-s) = \prod_{i=0}^{n} \left((n-s) - i \right)^{-b_{2i}} = \prod_{i=0}^{n} (-1)^{b_{2i}} \left(s - (n-i) \right)^{-b_{2i}} = (-1)^{\chi} \prod_{i=0}^{d} \left(s - (n-i) \right)^{-b_{2n-2i}} = (-1)^{\chi} \prod$$

where we used $b_{2n-2i} = b_{2i}$ in the last equation. If we substitute *i* by n - i in this expression, we obtain

$$\zeta_{\mathcal{X}}(n-s) = (-1)^{\chi} \prod_{i=0}^{n} (s-i)^{-b_{2i}} = (-1)^{\chi} \zeta_{\mathcal{X}}(s).$$

If *n* is odd, then there is an even number of non-trivial Betti numbers and $\chi = 2b_0 + 2b_2 + \cdots + 2b_{n-1}$ is even. If *n* is odd, then $\chi = 2b_0 + 2b_2 + \cdots + 2b_{n-2} + b_n$ has the same parity as b_n . Thus the additional statement. \Box

Remark 2. Note the similarity with the functional equation for motivic zeta functions as in [3, Theorem 1]. Amongst other factors, $(-1)^{\chi(M)}$ appears also in the functional equation of the zeta function of a motive M where $\chi(M)$ is the (positive part of the) Euler characteristic of M.

3. The functional equation for reductive groups over \mathbb{F}_1

The above observations imply further a functional equation for reductive group schemes over \mathbb{F}_1 . Note that Soulé's and Connes and Consani's approaches towards \mathbb{F}_1 -geometry indeed succeeded in descending split reductive group schemes from \mathbb{Z} to \mathbb{F}_1 (cf. [1,5,7]).

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Let *G* be a split reductive group scheme of rank *r* with Borel group *B* and maximal split torus $T \subset B$. Let *N* be the normalizer of *T* in *G* and $W = N(\mathbb{Z})/T(\mathbb{Z})$ be the Weyl group. The Bruhat decomposition of *G* (with respect to *T* and *B*) is the morphism

$$\coprod_{w\in W} BwB\to G,$$

induced by the subscheme inclusions $BwB \to G$, which has the property that it induces a bijection between the *k*-rational points for every field *k*. We have $B \simeq \mathbb{G}_m^r \times \mathbb{A}^N$ as schemes where *N* is the number of positive roots of *G*, and $BwB \simeq \mathbb{G}_m^r \times \mathbb{A}^{N+\lambda(w)}$ where $\lambda(w)$ is the length of $w \in W$. With this we can calculate the counting polynomial of *G* as

$$N(q) = \# \prod_{w \in W} BwB(\mathbb{F}_q) = (q-1)^r q^N \sum_{w \in W} q^{\lambda(w)}$$

The quotient variety G/B is a smooth projective scheme of dimension N with counting function $N_{G/B}(q) = ((q-1)^r q^N)^{-1} N(q) = \sum_{w \in W} q^{\lambda(w)}$. Let b_0, \ldots, b_{2N} be the Betti numbers of G/B, then we know from the previous section that $N_{G/B}(q) = \sum_{l=0}^{N} b_{2l}q^l$ and that $b_{2N-2l} = b_{2l}$.

Thus we obtain for the counting polynomial of G that

$$N(q) = q^{N} \left(\sum_{k=0}^{r} (-1)^{r-k} {r \choose k} q^{k} \right) \cdot \left(\sum_{l=0}^{N} b_{2l} q^{l} \right) = \sum_{i=0}^{d} \left(\sum_{k+l=i-N} (-1)^{r-k} {r \choose k} b_{2l} \right) q^{i}$$

where d = r + 2N is the dimension of *G* and with the convention that $\binom{r}{k} = 0$ if k < 0 or k > r. Denote by $a_i = \sum_{k+l=i-N} (-1)^{r-k} \binom{r}{k} b_{2l}$ the coefficients of N(q).

Lemma 3.1. We have $a_0 = \cdots = a_{N-1} = 0$ and $a_{d-i} = (-1)^r a_{i+N}$.

Proof. The first statement follows from the fact that N(q) is divisible by q^N . For the second statement we use the symmetries $\binom{r}{k} = \binom{r}{r-k}$ and $b_{2N-2l} = b_{2l}$ to calculate

$$a_{d-i} = \sum_{k+l=d-i-N} (-1)^{r-k} \binom{r}{k} b_{2l} = \sum_{k+l=d-i-N} (-1)^r (-1)^k \binom{r}{r-k} b_{2N-2l}.$$

When we substitute k by r - k and l by N - l in this equation and use d = r + 2N, we obtain

$$a_{d-i} = (-1)^r \sum_{k+l=(i+N)-N} (-1)^{r-k} \binom{r}{k} b_{2l},$$

which is the same as $(-1)^r a_{i+N}$. \Box

Suppose *G* has an elusive model \mathcal{G} over \mathbb{F}_1 . Then \mathcal{G} has the zeta function $\zeta_{\mathcal{G}}(s) = \prod_{i=0}^n (s-i)^{-a_i}$. Let $\chi = N(1) = \sum_{i=0}^d a_i$.

Theorem 3. The zeta function $\zeta_{\mathcal{G}}(s)$ satisfies the functional equation

$$\zeta_{\mathcal{G}}(r+N-s) = (-1)^{\chi} \left(\zeta_{\mathcal{G}}(s) \right)^{(-1)^r}.$$

Proof. We use the previous lemma and r + N = d - N to calculate that

$$\zeta_{\mathcal{G}}(r+N-s) = \prod_{i=0}^{n} (r+N-s-i)^{-a_i} = \prod_{i=0}^{n} (d-N-s-i)^{-(-1)^r a_{d-N-i}}.$$

After substituting *i* by d - N - i, we find that

$$\zeta_{\mathcal{G}}(r+N-s) = \prod_{i=0}^{n} (i-s)^{-(-1)^{r} a_{i}} = (-1)^{\sum a_{i}} \left(\prod_{i=0}^{n} (s-i)^{-a_{i}} \right)^{(-1)^{r}} = (-1)^{\chi} \left(\zeta_{\mathcal{G}}(s) \right)^{(-1)^{r}}.$$

Remark 4. Kurokawa calculates the \mathbb{F}_1 -zeta functions of \mathbb{P}^n , GL(n) and SL(n) in [4]. One can verify the functional equation for these examples immediately.

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