## Algebra/Algebraic Geometry

## Isotropy of symplectic involutions

## Isotropie d'involutions symplectiques

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## A R T I C L E I N F O

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#### Abstract

We prove the symplectic analogue of the isotropy theorem for orthogonal involutions. We apply (a modification of) a method due to J.-P. Tignol originally applied to prove the symplectic analogue of the hyperbolicity theorem for orthogonal involutions. © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

\section*{R É S U M É}

Nous démontrons l'analogue symplectique du théorème d'isotropie des involutions orthogonales. Nous utilisons (une modification de) la méthode due à J.-P. Tignol initialement utilisée pour démontrer l'analogue symplectique du théorème d'hyperbolicité des involutions orthogonales. © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.


We refer to [7] for terminology and basic facts concerning involutions on central simple algebras. Below, we'll meet myriads of finite odd degree field extensions; we simply call them odd for short.

In this Note we prove

Theorem 1. Let $F$ be a field of characteristic not 2, A a central simple $F$-algebra, $\sigma$ a symplectic involution on $A$. The following two conditions are equivalent:
(1) $\sigma$ becomes isotropic over any field extension $E / F$ such that ind $A_{E}=2$;
(2) $\sigma$ becomes isotropic over some odd extension of $F$.
(We recall that $\sigma$ is always isotropic and, moreover, hyperbolic as far as ind $A=1$.)
Theorem 1 is the symplectic analogue of the following result on orthogonal involutions:

Theorem 2. (See [5, Theorem 1].) Let $F$ be a field of characteristic not 2, A a central simple $F$-algebra, $\tau$ an orthogonal involution on $A$. The following two conditions are equivalent:
(1) $\tau$ becomes isotropic over any field extension $E / F$ such that ind $A_{E}=1$;
(2) $\tau$ becomes isotropic over some odd extension of $F$.

[^0]The symplectic analogue of an earlier and weaker than Theorem 2 result [4, Theorem 1.1] concerning hyperbolicity of orthogonal involution has been obtained by J.-P. Tignol in [8, Theorem 1]. We prove (the "difficult" part (1) $\Rightarrow$ (2) of) Theorem 1 by a slight modification of Tignol's method. The necessity of modification comes from the presence of odd extensions in the "isotropy business" and from their absence (due to [7, Corollary 6.16]) in the "hyperbolicity business". Note that a different modification, making use of valuations on quaternion skew fields, has been suggested by J.-P. Tignol himself. In contrast to this, our modification makes use of valuations on fields and is contained in Corollary 7, a statement on a field of Laurent series which has nothing to do with central simple algebras or involutions.

Let us explain the characteristic assumption char $F \neq 2$. Deducing Theorem 1 from Theorem 2, we need the characteristic assumption in order to reduce to a perfect base field, the need of a perfect field coming from Remark 8. Recall that anyway, the characteristic assumption is needed in the proof of Theorem 2 itself, because it exploits the Steenrod operations on the Chow groups modulo 2 which (the operations) are not available in characteristic 2 .

We are going to use several lemmas. The first one is elementary and easy, the others come from the classical theory of complete discrete valuation fields.

Lemma 3. (See [6, Lemma 3.3].) Let $F$ be a field, $K$ an odd extension of $F$, and $E$ an arbitrary field extension of $F$. Then there exists an odd extension $L / E$ and an $F$-embedding $K \hookrightarrow L$.

A coefficient field of a discrete valuation field $L$ is a subfield of the valuation ring of $L$ mapped under the residue map onto the residue field of $L$.

Lemma 4. Let $L$ be a complete discrete valuation field with characteristic 0 residue field, and let $F$ be a subfield of the valuation ring of $L$. Then $L$ has a coefficient field containing $F$.

Proof. Since the characteristic of the residue field is 0 (and $L$ is complete), any maximal subfield of the valuation ring of $L$ is a coefficient field [3, Proof of Proposition (5.2), Ch. II]. Therefore we may simply take a maximal subfield containing $F$.

Lemma 5. Let $L$ be a complete discrete valuation field and assume that $p:=$ char $L$ is a prime. Then
(1) L has a coefficient field;
(2) any coefficient field contains any perfect subfield of the valuation ring;
(3) if the residue of an element of the valuation ring is not a pth power, then this element is contained in some coefficient field.

Proof. (1) is [3, Proposition (5.4), Ch. II].
(2) is similar to [1, Theorem 10 (c)]. I order to prove (2), let us fix some coefficient field. Let $a$ be an element of a perfect subfield $F$ of the valuation ring, $b$ the image of $a$ under the residue map, and $c$ the element of the coefficient field mapped to $b$. Since $F$ is perfect, $a$ and $c$ are multiplicative representatives (also called Teichmüller representatives) of $b$ [3, definition in (7.1), Ch. I] (this notion makes sense only if the characteristic of the residue field is positive). Therefore $a=c$ by the uniqueness of the multiplicative representatives [3, Proposition (7.1), Ch. I].

To prove (3), let $b$ be an element of the residue field. If $b$ is not a $p$ th power, it can be included in a $p$-basis [3, definition in (5.3), Ch. II], of the residue field. Therefore, for any representative $a$ of $b$ (in the valuation ring), there exists a coefficient field containing $a$ [3, Proof of Proposition (5.4), Ch. II].

Corollary 6. Let $F$ be a perfect field, $x, t$ variables, and $\hat{L}$ an odd extension of the field $F((x))$. Then there exist a subfield $L \subset \hat{L}$ containing $F$ and odd over $F$, and an L-identification $L((t))=\hat{L}$ such that the product $x t$ is a square in $\hat{L}$.

Proof. We supply the field $\hat{L}$ with the (unique) extension $v$ of the $x$-adic valuation on $F((x))$. We are identifying the totally ordered group $v\left(\hat{L}^{\times}\right)$with $\mathbb{Z}$. The discrete valuation field $\hat{L}$ is complete [3, Theorem (2.5), Ch. II]. Let $L^{\prime}$ be its residue field. Then $L^{\prime}$ is a finite extension of $F$, moreover

$$
\left[L^{\prime}: F\right] \cdot v(x)=[\hat{L}: F((x))]
$$

[3, Theorem (2.5), Ch. II]. In particular, the integers $\left[L^{\prime}: F\right]$ and $v(x)$ are odd.
By Lemmas 4 and $5, \hat{L}$ has a coefficient field $L$ containing $F$. One can $L$-identify $\hat{L}$ with the field of Laurent series over $L$ in one variable corresponding to any given uniformizing element in $\hat{L}$ (that is, any element in $\hat{L}$ of valuation 1) [3, Corollary (5.2), Ch. I].

Let $s$ be a uniformizing element in $\hat{L}$ and set $t:=s^{v(x)+1} / x$. Then $t$ is also a uniformizing element, and $x t$ is a square in $\hat{L}$.

Corollary 7. Let $F$ be a perfect field, $x, y, t_{x}, t_{y}$ variables, and $\hat{L}$ an odd extension of the field $F((x))((y))$. Then there exist a subfield $L \subset \hat{L}$ containing $F$ and odd over $F$, and an L-identification $L\left(\left(t_{x}\right)\right)\left(\left(t_{y}\right)\right)=\hat{L}$ such that the products $x t_{x}$ and $y t_{y}$ are squares in $\hat{L}$.

Proof. We first consider the case where char $F=0$. In this case we simply apply Corollary 6 twice. Applying it first to the (perfect) field $F((x))$ and the odd extension $\hat{L} / F((x))((y))$, we get a subfield $\check{L} \subset \hat{L}$ containing $F((x))$ and odd over $F((x))$, and an $\check{L}$-identification $\check{L}\left(\left(t_{y}\right)\right)=\hat{L}$ such that $y t_{y}$ is a square in $\hat{L}$. Then we apply Corollary 6 for the second time, now to the field $F$ and the odd extension $\check{L} / F((x))$, getting this time a subfield $L \subset \check{L}$ containing $F$ and odd over $F$, and an $L$-identification $L\left(\left(t_{x}\right)\right)=\check{L}$ such that $x t_{x}$ is a square in $\check{L}$. Substituting, we get a required $L$-identification $L\left(\left(t_{x}\right)\right)\left(\left(t_{y}\right)\right)=\hat{L}$.

Now we assume that $p:=$ char $F>0$. Since the field $F((x))$ is no longer perfect, the above procedure has to be modified. The field $\hat{L}$ is complete with respect to the (unique) extension $v$ of the $y$-adic valuation on $F((x))((y))$. Let $\check{L}^{\prime}$ be its residue field. Then $\check{L}^{\prime}$ is a finite extension of $F((x))$ and

$$
\left[\check{L}^{\prime}: F((x))\right] \cdot v(y)=[\hat{L}: F((x))((y))] .
$$

In particular, the integers $\left[\check{L}^{\prime}: F((x))\right]$ and $v(y)$ are odd.
Applying Corollary 6 to the perfect field $F$ and the odd extension $\check{L}^{\prime} / F((x))$, we find a subfield $L^{\prime} \subset \check{L}^{\prime}$ containing $F$ and odd over $F$, and an $L^{\prime}$-identification $L^{\prime}\left(\left(t_{x}^{\prime}\right)\right)=\check{L}^{\prime}$ such that $x t_{x}^{\prime}$ is a square in $\check{L}^{\prime}: x t_{x}^{\prime}=b^{2}$ for some $b \in \check{L}^{\prime}$. Since $t_{x}^{\prime}$ is not a $p$ th power in $L^{\prime}\left(\left(t_{x}^{\prime}\right)\right)$, for an arbitrary chosen representative $t_{x}$ of $t_{x}^{\prime}$ in the valuation ring of $\hat{L}$ we can find by Lemma 5 a coefficient field of $\hat{L}$ containing $t_{x}$. Let $a$ be a representative of $b$. We choose $t_{\chi}:=a^{2} / x$ and write $\check{L}$ for a coefficient field containing this $t_{x}$. So, $\check{L}$ is a subfield of $\hat{L}$, and we can find an $\check{L}$-identification $\check{L}\left(\left(t_{y}\right)\right)=\hat{L}$ such that $y t_{y}$ is a square. Let $L$ be the subfield of the coefficient field $\check{L}$ corresponding to the subfield $L^{\prime}$ of the residue field $\check{L}^{\prime}$ of $\check{L}$. The field $L$ contains $F$ and is $F$-isomorphic to $L^{\prime}$; in particular, $L / F$ is odd. Furthermore, $\check{L}=L\left(\left(t_{x}\right)\right)$. Substituting, we get the identification $L\left(\left(t_{x}\right)\right)\left(\left(t_{y}\right)\right)=\hat{L}$. The product $x t_{x}$ is the square of $a \in \hat{L}$.

Remark 8. The statements of Corollaries 6 and 7 fail for general (imperfect) $F$.
Proof of Theorem 1. The implication $(2) \Rightarrow(1)$ is an easy consequence of the classical Springer theorem on quadratic forms [2, Corollary 18.5]. Assume that we are given an odd extension $L / F$ such that $\sigma_{L}$ is isotropic and a field extension $E / F$ such that ind $A_{E}=2$. By Lemma 3, there exists an odd extension $E L$ of $E$ containing $L$. Let $Q$ be a quaternion $E$-algebra Brauer-equivalent to $A_{E}$. We can find a right $Q$-module $V$, an isomorphism of $E$-algebras $E^{2} V \simeq A_{E}$, and a hermitian (with respect to the canonical involution on $Q$ ) form $h$ on $V$ such that the involution $\sigma_{E}$ is adjoint to $h$. Note that for any $v \in V$, the element $h(v, v) \in Q$ is symmetric and therefore lies in $E$ [7, Proposition (2.6)]. Let $q$ be the quadratic form on the vector $E$-space $V$ defined by $q(v)=h(v, v)$. We get the following chain of implications: $\sigma_{L}$ is isotropic $\Rightarrow \sigma_{E L}$ is isotropic $\Rightarrow h_{E L}$ is isotropic $\Rightarrow q_{E L}$ is isotropic $\Rightarrow$ (by the Springer theorem) $q$ is isotropic $\Rightarrow h$ is isotropic $\Rightarrow \sigma_{E}$ is isotropic.

The implication (1) $\Rightarrow(2)$ is proved by the method of [8] and with a help of Corollary 7. Since char $F \neq 2$, we may assume that $F$ is perfect (replacing an imperfect $F$ by its perfect closure). Let $\tilde{F}:=F(x, y)$ be the field of rational functions in variables $x$ and $y$ over $F$. Let now $Q$ be the quaternion $\tilde{F}$-algebra $(x, y)_{\tilde{F}}$. Let $\tilde{A}$ be the tensor product of the $\tilde{F}$-algebras $A_{\tilde{F}}$ and $Q$ endowed with the (orthogonal) involution $\tilde{\sigma}$ defined as the tensor product of $\sigma_{\tilde{F}}$ by the canonical involution on $Q$.

Let $\tilde{E}$ be the function field of the Severi-Brauer variety of $\tilde{A}$. Since the algebra $\tilde{A}_{\tilde{E}}$ is split, the algebra $A_{\tilde{E}}$ is Brauerequivalent to the quaternion algebra $Q_{\tilde{E}}$. In particular, ind $A_{\tilde{E}}$ divides 2 . It follows by (1) that the involution $\sigma_{\tilde{E}}$ is isotropic, i.e., $\sigma_{\tilde{E}}(a) \cdot a=0$ for some non-zero element $a \in A_{\tilde{E}}$. The element $b:=a \otimes 1 \in \tilde{A}_{\tilde{E}}$ is also non-zero and satisfies $\tilde{\sigma}_{\tilde{E}}(b) \cdot b=0$. Therefore the orthogonal involution $\tilde{\sigma}_{\tilde{E}}$ is isotropic. Applying Theorem 2, we get an odd extension $\tilde{L} / \tilde{F}$ such that the involution $\tilde{\sigma}_{\tilde{L}}$ is isotropic.

The field $\tilde{F}$ is a subfield of the field $\hat{F}:=F((x))((y))$. By Lemma 3, there exists an odd extension $\hat{L}$ of $\hat{F}$ containing $\tilde{L}$. The involution $\tilde{\sigma}_{\hat{L}}$ is isotropic for such $\hat{L}$. We apply Corollary 7, find the odd field extension $L / F$ and the identification $\hat{L}=L\left(\left(t_{x}\right)\right)\left(\left(t_{y}\right)\right)$. We note that the quaternion algebra $Q_{\hat{L}}=(x, y)_{\hat{L}}$ is isomorphic to $\left(t_{x}, t_{y}\right)_{\hat{L}}$ because $x t_{x}$ and $y t_{y}$ are squares. Now [8, Proposition 1] affirms that $\sigma_{L}$ is isotropic. This finishes the proof of Theorem 1.

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