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Algebraic Geometry

Note on local structure of Artin stacks

Note sur la structure locale de champs de Artin

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ARTICLE INFO	ABSTRACT
Article history: Received 8 July 2010 Accepted after revision 29 September 2010 Available online 12 October 2010 Presented by Christophe Soulé	In this Note we show that an Artin stack with finite inertia stack is étale locally isormorphic to the quotient of an affine scheme by an action of a general linear group. © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. R É S U M É
	Dans cette Note, nous montrons que tout champ algébrique dont l'inertie est finie, est étale-localement isomorphe au quotient d'un schéma affine par une action du groupe général linéaire. © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Let \mathcal{X} be an Artin stack over \mathbb{Z} (see [9]). A coarse moduli map (space) for \mathcal{X} is a morphism $\pi : \mathcal{X} \to X$ to an algebraic space X which has properties:

- π is universal among morphisms from \mathcal{X} to algebraic spaces, i.e., for any morphism $\phi : \mathcal{X} \to Y$ with an algebraic space Y there is a unique morphism $f : \mathcal{X} \to Y$ such that $\phi = f \circ \pi$,
- for any algebraically closed field K, π identifies the set of isomorphism classes of $\mathcal{X}(K)$ with the set X(K) of K-valued points of X.

Let \mathcal{X} be an Artin stack locally of finite type over a locally noetherian scheme *S*. Suppose that \mathcal{X} has finite inertia stack, that is, the projection

$$\mathrm{pr}_1: I_{\mathcal{X}} := \mathcal{X} \times_{\mathcal{X} \times_S \mathcal{X}} \mathcal{X} \to \mathcal{X}$$

is finite. In this setting, according to [7] there exist an algebraic space X locally of finite type over S and a coarse moduli map $\pi : \mathcal{X} \to X$ over S, where π is proper and quasi-finite. When in addition \mathcal{X} is a Deligne–Mumford stack, it has, étale locally on its coarse moduli space X, the form of a quotient stack [Z/G] where Z is an affine scheme and G is a finite group. This structure has been important and very useful in various situations. We would like to make the following useful observation:

Theorem 1. Let \mathcal{X} be an Artin stack locally of finite type over S with finite inertia stack. Then for any point $x \in \mathcal{X}$ there is an étale neighborhood $U \to X$ of $\pi(x)$ such that the S-stack $\mathcal{X} \times_X U$ has the form [Spec R/GL_n], where Spec R is an affine scheme over S and

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 GL_n is the general linear group scheme defined over S. Moreover, U is naturally isomorphic to the spectrum Spec R^{GL_n} of the invariant ring.

Proof. Let $\pi(x) = y$. Then according to the proof of the theorem of Keel and Mori (see [7] in particular Section 4, see also [3]), we see that there is an étale neighborhood $U \to X$ of y from an affine scheme U such that $\mathcal{X}_U := \mathcal{X} \times_X U$ has a finite flat surjective morphism $p: Y \to \mathcal{X}_U$ from a scheme Y. Then since Y is a scheme, $p_*\mathcal{O}_Y$ is a vector bundle on \mathcal{X}_U such that at each geometric point the stabilizer faithfully acts on the fiber. After shrinking \mathcal{X}_U if necessary we assume that \mathcal{X}_U is connected. Then by [4, 2.12] the total space of the corresponding principal GL_n-bundle W over \mathcal{X}_U is an algebraic space and thus \mathcal{X}_U is isomorphic to the quotient stack $[W/GL_n]$ where n is a non-negative integer. Now we will show that W is an affine scheme. Note that the projection $W \to [W/GL_n]$ is a GL_n-bundle and $\mathcal{X}_U \simeq [W/GL_n] \to U$ is proper. Hence W is separated and noetherian. In addition, $Y \to \mathcal{X}_U \to U$ is a finite morphism and thus Y is affine. Therefore $W \times_{\mathcal{X}_U} Y$ is affine because the second projection $W \times_{\mathcal{X}_U} Y \to Y$ is affine. Notice that the first projection $W \times_{\mathcal{X}_U} Y \to W$ is a finite surjective morphism. Consequently, applying Chevalley's theorem for algebraic spaces (see [8, III 4.1]) we deduce that W is affine. The last claim follows from the fact that $\mathcal{X}_U \to U$ is a coarse moduli map because $U \to X$ is étale. \Box

Remark 2. The action of GL_n on Spec R in Theorem 1 has a linearization. According to [14, Theorem 3.6, Corollary 3.7] there is a GL_n -equivariant closed immersion Spec $R \to \mathbb{A}_U^m$, where \mathbb{A}_U^m is an affine m-space over U which is endowed with a linear action of GL_n .

Remark 3. If \mathcal{X} is a Deligne–Mumford stack of finite type over *S* which has finite inertia stack, then étale locally on its coarse moduli space, \mathcal{X} is the quotient [Y/G] of an affine scheme *Y* by an action of a finite (constant) group *G* (see [1, 2.2.3]). If stabilizer group schemes at geometric points on \mathcal{X} are finite (not necessarily reduced) linearly reductive group schemes, in [2] it is shown that \mathcal{X} is, étale locally on its coarse moduli space, the quotient of an affine scheme by an action of a linearly reductive group scheme. In [5] such étale-local quotient structures was studied when \mathcal{X} has (not necessarily finite) linearly reductive stabilizers and satisfies the stability (see [5]). In these cases the stabilizer group schemes has no non-trivial deformation (i.e., has a unique deformation), and we may take *G* to be the stabilizer group scheme at a point on \mathcal{X} when we work over a field. This point is crucial for Luna's étale slice theorem. However, in positive characteristic case a general (finite) group scheme has many and rich deformations. Thus in general local structures of Artin stacks are not so simple as above cases. Indeed an Artin stack can contain the information arising from non-trivial flat deformations of *G*, i.e., *BG*.

The typical usage of Theorem 1 is the reduction of problems to the case of group actions. In the rest of this note, we will present one of such applications of Theorem 1, which is a direct one. For this we shall prepare our setup.

Let \mathcal{X} be an Artin stack of finite type over \mathbb{Z} . We will denote by $G(\mathcal{X})$ (resp. $K(\mathcal{X})$) the algebraic K-theory spectrum of the exact category of coherent sheaves (resp. vector bundles) on \mathcal{X} , and we let $G(\mathcal{X}) \otimes \mathbb{Q}$ and $K(\mathcal{X}) \otimes \mathbb{Q}$ Bousfield localizations of $G(\mathcal{X})$ and $K(\mathcal{X})$ respectively, with respect to \mathbb{Q} (see [6]). Let us recall the isovariant étale descent of *G*-theory due to Joshua [6], which generalizes Thomason's descent [12]. Joshua generalized the notion of isovariant étale morphisms [15] in the equivariant case to arbitrary stacks and proved the descent theorem. We can state our result without using terms of isovariant étale morphisms, but we recall the notion for the reader's convenience. A morphism $\mathcal{Y} \to \mathcal{X}$ of Artin stacks is isovariant if $I_{\mathcal{Y}} \to I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{Y}$ is an isomorphism, where $I_{\mathcal{X}}$ and $I_{\mathcal{Y}}$ denote inertia stacks of \mathcal{X} and \mathcal{Y} respectively. Note that an isovariant morphism is representable. If $\mathcal{X} \to Z$ is a morphism to an algebraic space Z and $W \to Z$ is a morphism of algebraic spaces, then the projection $\mathcal{X} \times_Z W \to \mathcal{X}$ is isovariant. Conversely, if \mathcal{X} has finite inertia stack and $\pi : \mathcal{X} \to X$ is a coarse moduli map, then for any isovariant étale morphism $\mathcal{Y} \to \mathcal{X}$ there exists an étale morphism $Z \to X$ such that $\pi^{-1}(Z) \to \mathcal{X}$ is naturally isomorphic to $\mathcal{Y} \to \mathcal{X}$ (cf. [15, 2.17]). Indeed if we denote by Y a coarse moduli space for \mathcal{Y} and by $Y \to X$ the induced morphism, then by [3, 4.2] $Y \to X$ is étale (it suffices to check this locally on X). Thus it suffices to see that the natural étale morphism $\mathcal{Y} \to \pi^{-1}(Y)$ is an isomorphism. Note that $\mathcal{Y} \to \pi^{-1}(Y)$ is separated. To check this we may assume that Y is affine and $\pi^{-1}(Y)$ has a finite flat cover $W \to \pi^{-1}(Y)$ by an affine scheme W. Recall that $\mathcal{Y} \to Y$ and $\pi^{-1}(Y) \to Y$ are proper and quasi-finite. Then the natural morphism $W \times_{\pi^{-1}(Y)} \mathcal{Y} \to \mathcal{Y} \to Y$ is finite, and we conclude that $W \times_{\pi^{-1$

We continue to assume that \mathcal{X} has finite inertia stack and let $\pi : \mathcal{X} \to X$ be a coarse moduli map. Then using Quillen's Q-construction and the loop functor, we have two presheaves of spectra

$$\mathbf{G}: (X_{et})^{op} \to \mathbf{Spt}$$

and

$$\mathbf{K}: (X_{et})^{op} \to \mathbf{Spt}$$

which to any $Y \to X$ in the étale site X_{et} associate $G(\pi^{-1}(Y))$ and $K(\pi^{-1}(Y))$ respectively, where **Spt** is the category of spectra. For a presheaf of spectra *P* let $\mathbb{H}_{et}(X, P)$ be the hypercohomology with respect to étale topology on *X*, defined

Proposition 4 (Poincaré duality). Let \mathcal{X} be a regular Artin stack of finite type over \mathbb{Z} with finite inertia stack. Let $\pi : \mathcal{X} \to \mathcal{X}$ be a coarse moduli map. Then the natural map

 $\mathbb{H}_{et}(X, \mathbf{K} \otimes \mathbb{Q}) \to \mathbb{H}_{et}(X, \mathbf{G} \otimes \mathbb{Q})$

is a weak equivalence of spectra.

Proof. To show our claim, clearly we may work étale locally on the coarse moduli space *X*. Thus according to Theorem 1 we may and will assume that \mathcal{X} is of the form [Spec R/GL_n]. Now we can apply the result of Thomason [13, Theorem 5.7] to obtain our proposition. \Box

Remark 5. Proposition 4 also holds for other localized G-theories [6, 5.1.5].

Acknowledgement

We would like to thank the referee for making improvements in Remark 2.

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