Numerical Analysis

# Error estimates of the discretization of linear parabolic equations on general nonconforming spatial grids 

# Des estimations d'erreurs pour la discrétisation des équations paraboliques sur une classe générale multidimensionnelle de maillages non conformes 

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## A R T I C L E IN F O

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#### Abstract

A general class of nonconforming meshes has been recently used to approximate stationary anisotropic heterogeneous diffusion problems in any space dimensions. The aim of the present work is to deal with some error estimates of the discretization of parabolic equations on this general class of meshes in several space dimensions. We present an implicit scheme based on an orthogonal projection of the exact initial function. We provide error estimates in discrete norms $\mathbb{L}^{\infty}\left(0, T ; H_{0}^{1}(\Omega)\right)$ and $\mathcal{W}^{1, \infty}\left(0, T ; \mathbb{L}^{2}(\Omega)\right)$. © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. R É S U M É Une classe assez générale de maillages non conformes a été récemment utilisée pour approximer les équations stationnaires de diffusion hétérogène anisotrope pour toute dimension d'espace. Le but de ce travail est d'obtenir des estimations d'erreur pour la discrétisation des équations paraboliques sur cette classe générale de maillages. On présente un schéma implicite où la condition initiale a été discrétisée en utilisant une "projection orthogonale" de la condition initiale. Nous fournissons des estimations d'erreur en normes discrètes de $\mathbb{L}^{\infty}\left(0, T ; H_{0}^{1}(\Omega)\right)$ et $\mathcal{W}^{1, \infty}\left(0, T ; \mathbb{L}^{2}(\Omega)\right)$.


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## 1. Problem to be solved and discrete data

The present work deals with the following multidimensional transient diffusion problem:

$$
\begin{equation*}
u_{t}(x, t)-\Delta u(x, t)=f(x, t), \quad(x, t) \in \Omega \times(0, T) \tag{1}
\end{equation*}
$$

where $\Omega$ is an open bounded polyhedral subset in $\mathbb{R}^{d}$, with boundary $\partial \Omega=\bar{\Omega} \backslash \Omega, d \in \mathbb{N}^{\star}, T>0$, and $f$ is a given function. Initial and boundary conditions are given by

$$
\begin{align*}
& u(x, 0)=u^{0}(x), \quad x \in \Omega  \tag{2}\\
& u(x, t)=0, \quad(x, t) \in \partial \Omega \times(0, T) \tag{3}
\end{align*}
$$

[^0]

Fig. 1. Notations for two neighboring control volumes in the case $d=2$.

The discretization of $\Omega$ is performed using the mesh $\mathcal{D}=(\mathcal{M}, \mathcal{E}, \mathcal{P})$ described in [1, Definition 2.1]. An example of two neighboring control volumes $K$ and $L$ of $\mathcal{M}$ is depicted in Fig. 1. The time discretization is performed with a constant time step $k=\frac{T}{N+1}$, where $N \in \mathbb{N}^{\star}$, and we shall denote $t_{n}=n k$, for $n \in \llbracket 0, N+1 \rrbracket$. Throughout this Note, the letter $C$ stands for a positive constant independent of the parameters of the space and time discretizations.

Next, we shall use the same notations as in [1]. In particular, $\mathcal{X}_{\mathcal{D}}$ is the set of all $\left(\left(v_{K}\right)_{K \in \mathcal{M}},\left(v_{\sigma}\right)_{\sigma \in \mathcal{E}}\right)$, and $\mathcal{X}_{\mathcal{D}, 0} \subset \mathcal{X}_{\mathcal{D}}$ is the set of all $v \in \mathcal{X}_{\mathcal{D}}$ such that $v_{\sigma}=0$ for all $\sigma \in \mathcal{E}_{\text {ext }}$. In order to analyze the convergence, we need to consider the size of the discretization $\mathcal{D}$ defined by $h_{\mathcal{D}}=\sup \{\operatorname{diam}(K), K \in \mathcal{M}\}$ and the regularity of the mesh given by $\theta_{\mathcal{D}}=$ $\max \left(\max _{\sigma \in \mathcal{E}_{\text {int }}, K, L \in \mathcal{M}} \frac{d_{K, \sigma}}{d_{L, \sigma}}, \max _{K \in \mathcal{M}, \sigma \in \mathcal{E}_{K}} \frac{h_{K}}{d_{K, \sigma}}\right)$.

## 2. The finite volume scheme and its convergence order

The scheme is based on the discrete gradient given in [1]: for $u \in \mathcal{X}_{\mathcal{D}}$, we define, for all $K \in \mathcal{M}$

$$
\begin{equation*}
\nabla_{\mathcal{D}} u(x)=\nabla_{K, \sigma} u, \quad \text { a.e. } x \in \mathcal{D}_{K, \sigma}, \tag{4}
\end{equation*}
$$

where $\mathcal{D}_{K, \sigma}$ is the cone with vertex $x_{K}$ and basis $\sigma$ and

$$
\begin{equation*}
\nabla_{K, \sigma} u=\frac{1}{\mathrm{~m}(K)} \sum_{\sigma \in \mathcal{E}_{K}} \mathrm{~m}(\sigma)\left(u_{\sigma}-u_{K}\right) \mathbf{n}_{K, \sigma}+\left(\frac{\sqrt{d}}{d_{K, \sigma}}\left(u_{\sigma}-u_{K}-\nabla_{K} u \cdot\left(x_{\sigma}-x_{K}\right)\right)\right) \mathbf{n}_{K, \sigma}, \tag{5}
\end{equation*}
$$

where $\nabla_{K} u=\frac{1}{\mathrm{~m}(K)} \sum_{\sigma \in \mathcal{E}_{K}} \mathrm{~m}(\sigma)\left(u_{\sigma}-u_{K}\right) \mathbf{n}_{K, \sigma}$ and $\mathbf{n}_{K, \sigma}$ is the unit vector normal to $\sigma$ outward to $K$.
First, find $u_{\mathcal{D}}^{0} \in \mathcal{X}_{\mathcal{D}, 0}$ such that

$$
\begin{equation*}
\left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^{0}, \nabla_{\mathcal{D}} v\right)_{\left(\mathbb{L}^{2}(\Omega)\right)^{d}}=-\left(\Delta u^{0}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^{2}(\Omega)}, \quad \forall v \in \mathcal{X}_{\mathcal{D}, 0} \tag{6}
\end{equation*}
$$

then, for and any $n \in \llbracket 0, N \rrbracket$, find $u_{\mathcal{D}}^{n+1} \in \mathcal{X}_{\mathcal{D}, 0}$ such that

$$
\begin{equation*}
\left(\partial^{1} \Pi_{\mathcal{M}} u_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^{2}(\Omega)}+\left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^{n+1}, \nabla_{\mathcal{D}} v\right)_{\left(\mathbb{L}^{2}(\Omega)\right)^{d}}=\sum_{K \in \mathcal{M}} \mathrm{~m}(K) f_{K}^{n} v_{K}, \quad \forall v \in \mathcal{X}_{\mathcal{D}, 0} \tag{7}
\end{equation*}
$$

where $\partial^{1} v^{n}=\frac{v^{n}-v^{n-1}}{k}, f_{K}^{n}=\frac{1}{k m(K)} \int_{t_{n}}^{t_{n+1}} \int_{K} f(x, t) \mathrm{d} x \mathrm{~d} t$, and $(\cdot, \cdot)_{\mathbb{L}^{2}(\Omega)}$ (resp. $(\cdot, \cdot)_{\left.\left(\mathbb{L}^{2}(\Omega)\right)^{d}\right)}$ denotes the $\mathbb{L}^{2}(\Omega)$ (resp. $\left(\mathbb{L}^{2}(\Omega)\right)^{d}$ ) inner product.

Remark 1. The choice of the discretization (6) of the initial condition (2) is useful in the second step of the proof of Theorem 2.1. Indeed, taking $n=0$ in (12) and using (2) with (6) implies that $\eta_{\mathcal{D}}^{0}=0$, where $\eta_{\mathcal{D}}^{n}=\bar{u}_{\mathcal{D}}^{n}-u_{\mathcal{D}}^{n}$ for all $n \in$ $\llbracket 0, N+1 \rrbracket$, which will allow to obtain (18) for the first time step.

Theorem 2.1. Let $\Omega$ be a polyhedral open bounded subset of $\mathbb{R}^{d}$, where $d \in \mathbb{N} \backslash\{0\}$, and $\partial \Omega=\bar{\Omega} \backslash \Omega$ its boundary. Assume that the solution of (1)-(3) satisfies $u \in \mathcal{C}^{2}\left([0, T] ; \mathcal{C}^{2}(\bar{\Omega})\right)$. Let $k=\frac{T}{N+1}$, with $N \in \mathbb{N}^{\star}$, and denote by $t_{n}=n k$, for $n \in \llbracket 0, N+1 \rrbracket$. Let $\mathcal{D}=(\mathcal{M}, \mathcal{E}, \mathcal{P})$ be a discretization in the sense of $\left[1\right.$, Definition 2.1]. Assume that $\theta_{\mathcal{D}}$ satisfies $\theta \geqslant \theta_{\mathcal{D}}$. Then there exists a unique solution $\left(u_{\mathcal{D}}^{n}\right)_{n \in \llbracket 0, N+1 \rrbracket}$ for (7)-(6). For each $n \in \llbracket 0, N+1 \rrbracket$, let us define the error $e_{\mathcal{M}}^{n} \in H_{\mathcal{M}}(\Omega)$ by:

$$
\begin{equation*}
e_{\mathcal{M}}^{n}=\mathcal{P}_{\mathcal{M}} u\left(\cdot, t_{n}\right)-\Pi_{\mathcal{M}} u_{\mathcal{D}}^{n} \tag{8}
\end{equation*}
$$

Then, the following error estimates hold:

- discrete $\mathbb{L}^{\infty}\left(0, T ; H_{0}^{1}(\Omega)\right)$-estimate: for all $n \in \llbracket 0, N+1 \rrbracket$

$$
\begin{equation*}
\left\|e_{\mathcal{M}}^{n}\right\|_{1,2, \mathcal{M}} \leqslant C\left(h_{\mathcal{D}}+k\right)\|u\|_{\mathcal{C}^{2}\left([0, T] ; \mathcal{C}^{2}(\bar{\Omega})\right)} \tag{9}
\end{equation*}
$$

- $\mathcal{W}^{1, \infty}\left(0, T ; \mathbb{L}^{2}(\Omega)\right)$-estimate: for all $n \in \llbracket 1, N+1 \rrbracket$

$$
\begin{equation*}
\left\|\partial^{1} e_{\mathcal{M}}^{n}\right\|_{\mathbb{L}^{2}(\Omega)} \leqslant C\left(h_{\mathcal{D}}+k\right)\|u\|_{\mathcal{C}^{2}\left([0, T] ; \mathcal{C}^{2}(\bar{\Omega})\right)} \tag{10}
\end{equation*}
$$

- error estimate in the gradient approximation: for all $n \in \llbracket 0, N+1 \rrbracket$

$$
\begin{equation*}
\left\|\nabla_{\mathcal{D}} u_{\mathcal{D}}^{n}-\nabla u\left(\cdot, t_{n}\right)\right\|_{\left(\mathbb{L}^{2}(\Omega)\right)^{d}} \leqslant C\left(h_{\mathcal{D}}+k\right)\|u\|_{\mathcal{C}^{2}\left([0, T] ; \mathcal{C}^{2}(\bar{\Omega})\right)} \tag{11}
\end{equation*}
$$

Sketch of the proof of Theorem 2.1. The uniqueness of ( $\left.u_{\mathcal{D}}^{n}\right)_{n \in \llbracket 0, N+1 \rrbracket}$ satisfying (7)-(6) can be deduced from the stability [1, (37), Lemma 4.1]. As usual, we can use this uniqueness to prove the existence.

To prove (9)-(11), we compare the solution $\left(u_{\mathcal{D}}^{n}\right)_{n \in \llbracket 0, N+1 \rrbracket}$ satisfying (7)-(6) with the solution defined by: for any $n \in$ $\llbracket 0, N+1 \rrbracket$, find $\bar{u}_{\mathcal{D}}^{n} \in \mathcal{X}_{\mathcal{D}, 0}$ such that

$$
\begin{equation*}
\left(\nabla_{\mathcal{D}} \bar{u}_{\mathcal{D}}^{n}, \nabla_{\mathcal{D}} v\right)_{\left(\mathbb{L}^{2}(\Omega)\right)^{d}}=-\sum_{K \in \mathcal{M}} v_{K} \int_{K} \Delta u\left(x, t_{n}\right) \mathrm{d} x, \quad \forall v \in \mathcal{X}_{\mathcal{D}, 0} \tag{12}
\end{equation*}
$$

Step 1. (Comparison between $u$ and $\bar{u}_{\mathcal{D}}^{n}$.) Thanks to the proof of [1, Theorem 4.2], we can prove that

$$
\begin{equation*}
\left\|\mathcal{P}_{\mathcal{M}} u\left(\cdot, t_{n}\right)-\Pi_{\mathcal{M}} \bar{u}_{\mathcal{D}}^{n}\right\|_{1,2, \mathcal{M}} \leqslant C h_{\mathcal{D}}\|u\|_{\mathcal{C}\left([0, T] ; \mathcal{C}^{2}(\bar{\Omega})\right)} \tag{13}
\end{equation*}
$$

We remark that, for all $j \in \llbracket 0,2 \rrbracket$, for all $n \in \llbracket j, N+1 \rrbracket$

$$
\begin{equation*}
\left(\nabla_{\mathcal{D}} \partial^{j} \bar{u}_{\mathcal{D}}^{n}, \nabla_{\mathcal{D}} v\right)_{\left(\mathbb{L}^{2}(\Omega)\right)^{d}}=-\sum_{K \in \mathcal{M}} v_{K} \int_{K} \Delta \partial^{j} u\left(x, t_{n}\right) \mathrm{d} x, \quad \forall v \in \mathcal{X}_{\mathcal{D}, 0} \tag{14}
\end{equation*}
$$

where we have denoted $\partial^{0} v^{n}=v^{n}$ and $\partial^{2} v^{n}=\frac{1}{k}\left(\partial^{1} v^{n}-\partial^{1} v^{n-1}\right)$. Therefore, thanks to [1, (62), Theorem 4.2] and the proof of [1, (64), Theorem 4.2], we can prove that for all $n \in \llbracket j, N+1 \rrbracket$

$$
\begin{equation*}
\left\|\partial^{j}\left(\mathcal{P}_{\mathcal{M}} u\left(\cdot, t_{n}\right)-\Pi_{\mathcal{M}} \bar{u}_{\mathcal{D}}^{n}\right)\right\|_{\mathbb{L}^{2}(\Omega)} \leqslant C h_{\mathcal{D}}\|u\|_{\mathcal{C}^{j}\left([0, T] ; \mathcal{C}^{2}(\bar{\Omega})\right)} \tag{15}
\end{equation*}
$$

Using the proof of [1, (63)-(64), Theorem 4.2], we prove that

$$
\begin{equation*}
\left\|\nabla_{\mathcal{D}} \bar{u}_{\mathcal{D}}^{n}-\nabla u\left(\cdot, t_{n}\right)\right\|_{\left(\mathbb{L}^{2}(\Omega)\right)^{d}} \leqslant C h_{\mathcal{D}}\|u\|_{\mathcal{C}^{2}\left([0, T] ; \mathcal{C}^{2}(\bar{\Omega})\right)} . \tag{16}
\end{equation*}
$$

Step 2. (Comparison between $\bar{u}_{\mathcal{D}}^{n}$ and $u_{\mathcal{D}}^{n}$.) Let $\eta_{\mathcal{D}}^{n}=\bar{u}_{\mathcal{D}}^{n}-u_{\mathcal{D}}^{n}$; from (12) and (7), we get

$$
\begin{align*}
& \left(\partial^{1} \Pi_{\mathcal{M}} \eta_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^{2}(\Omega)}+\left(\nabla_{\mathcal{D}} \eta_{\mathcal{D}}^{n+1}, \nabla_{\mathcal{D}} v\right)_{\left.\mathbb{L}^{2}(\Omega)\right)^{d}} \\
& \quad=-\sum_{K \in \mathcal{M}}\left(\mathrm{~m}(K) f_{K}^{n}+\int_{K} \Delta u\left(x, t_{n+1}\right) \mathrm{d} x\right) v_{K}+\left(\partial^{1} \Pi_{\mathcal{M}} \bar{u}_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v\right)_{\mathbb{L}^{2}(\Omega)}, \quad \forall v \in \mathcal{X}_{\mathcal{D}, 0} \tag{17}
\end{align*}
$$

Setting $n=0$ in (17), substituting $f$ by $u_{t}-\Delta u$ (subject of (1)), using suitable Taylor expansions, the Cauchy-Schwarz inequality, and (15) (when $j=1$ ), and taking $v=\partial^{1} \eta_{\mathcal{D}}^{1}$, (17) leads to (recall that $\partial^{1} \eta_{\mathcal{D}}^{1}=\frac{\eta_{\mathcal{D}}^{1}-\eta_{\mathcal{D}}^{0}}{k}=\frac{\eta_{\mathcal{D}}^{1}}{k}$ since $\eta_{\mathcal{D}}^{0}=0$ )

$$
\begin{equation*}
\left\|\partial^{1} \Pi_{\mathcal{M}} \eta_{\mathcal{D}}^{1}\right\|_{\mathbb{L}^{2}(\Omega)} \leqslant C\left(h_{\mathcal{D}}+k\right)\|u\|_{\mathcal{C}^{2}\left([0, T] ; \mathcal{C}^{2}(\bar{\Omega})\right)} \tag{18}
\end{equation*}
$$

Acting the discrete operator $\partial^{1}$ on the both sides of the previous equality, we get, for all $n \in \llbracket 1, N \rrbracket$

$$
\begin{align*}
& \left(\partial^{2} \Pi_{\mathcal{M}} \eta_{\mathcal{D}}^{n+1}, \Pi_{\mathcal{M}} v\right)_{L^{2}(\Omega)}+\left(\partial^{1} \nabla_{\mathcal{D}} \eta_{\mathcal{D}}^{n+1}, \nabla_{\mathcal{D}} v\right)_{\left(\mathbb{L}^{2}(\Omega)\right)^{d}} \\
& \quad=-\sum_{K \in \mathcal{M}}\left(\mathrm{~m}(K) \partial^{1} f_{K}^{n}+\int_{K} \Delta \partial^{1} u\left(x, t_{n+1}\right) \mathrm{d} x\right) v_{K}+\left(\partial^{2} \Pi_{\mathcal{M}} \bar{u}^{n+1}, \Pi_{\mathcal{M}} v\right)_{L^{2}(\Omega)}, \quad \forall v \in \mathcal{X}_{\mathcal{D}, 0} . \tag{19}
\end{align*}
$$

Taking $v=\partial^{1} \eta_{\mathcal{D}}^{n+1}$ in (19), substituting $f$ by $u_{t}-\Delta u$, using suitable Taylor expansions, and using (15) (when $j=2$ ), (19) leads to, recall that $\partial^{2} \eta_{\mathcal{D}}^{n+1}=\left(\partial^{1} \eta_{\mathcal{D}}^{n+1}-\partial^{1} \eta_{\mathcal{D}}^{n}\right) / k,\left\|\partial^{1} \Pi_{\mathcal{M}} \eta_{\mathcal{D}}^{n+1}\right\|_{\mathbb{L}^{2}(\Omega)} \leqslant\left\|\partial^{1} \Pi_{\mathcal{M}} \eta_{\mathcal{D}}^{n}\right\|_{\mathbb{L}^{2}(\Omega)}+C k\left(h_{\mathcal{D}}+k\right)\|u\|_{\mathcal{C}^{2}\left([0, T] ; \mathcal{C}^{2}(\bar{\Omega})\right)}$. This with (18) implies that (recall that $\eta_{\mathcal{D}}^{0}=0$ )

$$
\begin{equation*}
\left\|\partial^{1} \Pi_{\mathcal{M}} \eta_{\mathcal{D}}^{n}\right\|_{\mathbb{L}^{2}(\Omega)} \leqslant C\left(h_{\mathcal{D}}+k\right)\|u\|_{\mathcal{C}^{2}\left([0, T] ; \mathcal{C}^{2}(\bar{\Omega})\right)}, \quad \forall n \in \llbracket 0, N+1 \rrbracket . \tag{20}
\end{equation*}
$$

This with (15) yields (10). Substituting $f$ by $u_{t}-\Delta u$ in (17), using suitable Taylor expansions, and taking $v=\eta_{\mathcal{D}}^{n+1}$ in the resulting equation; using the Cauchy-Schwarz inequality, [1, (75), Lemma 5.3] when $p=2$, [1, (37), Lemma 4.1], (20), (15) (when $j=1$ ), and the fact that $\eta_{\mathcal{D}}^{0}=0$ implies that

$$
\begin{equation*}
\left|\eta_{\mathcal{D}}^{n}\right|_{\mathcal{X}} \leqslant C\left(h_{\mathcal{D}}+k\right)\|u\|_{\mathcal{C}^{2}\left([0, T] ; \mathcal{C}^{2}(\bar{\Omega})\right)}, \quad \forall n \in \llbracket 0, N+1 \rrbracket, \tag{21}
\end{equation*}
$$

this with [1, (36)] yields that, for all $n \in \llbracket 0, N+1 \rrbracket$

$$
\begin{equation*}
\left\|\Pi_{\mathcal{M}} \eta_{\mathcal{D}}^{n}\right\|_{1,2, \mathcal{M}} \leqslant C\left(h_{\mathcal{D}}+k\right)\|u\|_{\mathcal{C}^{2}\left([0, T] ; \mathcal{C}^{2}(\bar{\Omega})\right)} \tag{22}
\end{equation*}
$$

This with (13) leads to (9). Gathering estimates (21) and [1, (37), Lemma 4.1] yields $\left\|\nabla_{\mathcal{D}} \eta_{\mathcal{D}}^{n}\right\|_{\left(\mathbb{L}^{2}(\Omega)\right)^{d}} \leqslant C\left(h_{\mathcal{D}}+k\right)$. $\|u\|_{\mathcal{C}^{2}\left([0, T] ; \mathcal{C}^{2}(\bar{\Omega})\right)}$, and then we combine this with (16) to get (11).

## References

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## Further reading

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