

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I



www.sciencedirect.com

Mathematical Analysis/Functional Analysis

(e)-convergence and related problem

(e)-convergence et problème connexe

Mübariz Tapdıgoğlu Karaev

Isparta Vocational School, Suleyman Demirel University, 32260, Isparta, Turkey

ARTICLE INFO

Article history: Received 23 May 2010 Accepted after revision 16 September 2010 Available online 15 October 2010

Presented by Gilles Pisier

ABSTRACT

We answer negatively to a question of Zorboska (2003) [13], which is concerned to the boundary behavior of Berezin symbols of Bergman space operators. We also introduce the notions of (e)-summability of sequences and series of complex numbers, and study some of their properties. As a corollary, we obtain the classical Abel theorems of summability theory.

© 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

RÉSUMÉ

On donne une réponse négative à une question posée par Zobroska (2003) dans [13] ; cette question porte sur le comportement à la frontière des symboles de Berezin d'opérateurs spaciaux de Bergman. On introduit aussi les notions de (*e*)-sommabilité de suites et de séries de nombres complexes et on étudie certaines de leurs propriétés. Comme corollaire, on retrouve les théorèmes classiques de Abel sur la théorie de la sommabilité.

© 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

Berezin [2,3] introduces the notion of covariant and contravariant symbols of an operator. Berger and Coburn [4,5] are the first to actually use the contravariant symbol of a Toeplitz operator, the so-called Berezin symbol.

In this article by applying summability theory we solve a problem posed by Zorboska in [13], which is concerned to the boundary behavior of Berezin symbols of Bergman space operators. Namely, we give a concrete negative answer to a question of Zorboska, while in [10] we have proposed a general procedure for constructing such examples.

Recall that the Bergman space $L_a^2 = L_a^2(\mathbb{D})$ is the Hilbert space of all analytic functions f on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for which

$$\|f\|_{L^2_a} = \left(\int_{\mathbb{D}} \left|f(z)\right|^2 \mathrm{d}A(z)\right)^{1/2} < \infty,$$

where $dA = \frac{dx dy}{\pi}$ is the normalized area measure on \mathbb{D} . If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are two functions in L^2_a , then the inner product of f and g is given by

E-mail address: garayev@fef.sdu.edu.tr.

¹⁶³¹⁻⁰⁷³X/\$ – see front matter © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. doi:10.1016/j.crma.2010.09.017

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) \overline{g(z)} \, \mathrm{d}A(z) = \sum_{n=0}^{\infty} \frac{a_n \overline{b_n}}{n+1}.$$

The Berezin symbol of a bounded linear operator T on the Bergman space L_a^2 is the function

$$\widetilde{T}(\lambda) := \left\langle T \frac{k_{\lambda}}{\|k_{\lambda}\|}, \frac{k_{\lambda}}{\|k_{\lambda}\|} \right\rangle \quad (\lambda \in \mathbb{D})$$

where $k_{\lambda}(z) = \frac{1}{(1-\bar{\lambda}z)^2}$ is the reproducing kernel of L_a^2 .

It is known that on the most familiar functional Hilbert spaces, including the spaces L_a^2 and $H^2 = H^2(\mathbb{D})$ (Hardy space), the Berezin symbol uniquely determines the operator (i.e., $\tilde{T}_1(\lambda) = \tilde{T}_2(\lambda)$ for all λ implies $T_1 = T_2$), see for instance Fricain [7]. Thus, the Berezin symbol is very effective in many cases in the sense that it contains a lot of information about the operator that induces it. Successful applications of the Berezin symbol are so far mainly in the study of Hankel and Toeplitz operators [12]. This method is motivated by its connections with quantum physics (see, for instance, [2,3]). More informations about Berezin symbols can be found, for instance in [9,11,12].

It is known more about the boundary values of the Berezin symbol of a compact operator T. Since $\{\frac{k_{\lambda}}{\|k_{\lambda}\|}\}$ converges weakly and uniformly to zero as |z| converges to 1, $\tilde{T}(\lambda)$ converges to zero uniformly as |z| approaches 1, whenever the operator T is compact on L_a^2 . One of the deeper results on Berezin symbols obtained in [1] and [6] (see also [9, Theorem 2.11]) states that a function φ is a bounded harmonic function on \mathbb{D} if and only if $\varphi(\lambda) = \tilde{T}_{\varphi}(\lambda)$ for every $\lambda \in \mathbb{D}$; here T_{φ} is the Toeplitz operator on L_a^2 . Note that bounded harmonic functions have radial boundary values almost everywhere on the unit circle $\mathbb{T} = \partial \mathbb{D}$. It is also true that if φ is a continuous function on the closed unit disc $\overline{\mathbb{D}}$, then \tilde{T}_{φ} is in $C(\overline{\mathbb{D}})$ and $\tilde{T}_{\varphi} = \varphi$ on \mathbb{T} (see Zhu [12, Proposition 6.1.6]). In this connection, in [13], Zorboska formulated the following natural and fundamental question: does the Berezin symbol of a bounded operator on $L_a^2(\mathbb{D})$ have radial limits almost everywhere on the unit circle \mathbb{T} ?

This article, in particular, answers this question negatively. Namely, we give a concrete example to a bounded diagonal operator on the Bergman space L_a^2 such that its Berezin symbol has no radial limits even anywhere on the unit circle \mathbb{T} (see Theorem 2.1 below). The diagonal operators technique is also used in the study of (*e*)-summability, which is introduced in Section 2 (see Theorem 3.2).

Before giving our results, we note that the Berezin symbol of an operator T on the Bergman space L_a^2 has an explicit formula given by

$$\widetilde{T}(\lambda) = \left(1 - |\lambda|^2\right)^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sqrt{(n+1)(m+1)} \langle T z^n, z^m \rangle \overline{\lambda}^n \lambda^m,$$
(1)

for every $\lambda \in \mathbb{D}$.

2. A counterexample to Zorboska's question

The following result gives negative answer to Zorboska's question:

Theorem 2.1. Let $a_n = n^{-ic}$, where $c \in \mathbb{R} \setminus \{0\}$, and let $b_n := \frac{1}{n+1} \sum_{j=0}^n a_j$, $n \ge 0$ (we put $a_0 := 0$). Let $D_{\{b_n\}}$ be the diagonal operator with diagonal elements b_n with respect to the standard orthonormal basis $e_n(z) = \sqrt{n+1}z^n$, $n \ge 0$, of the Bergman space L_a^2 . Then the Berezin symbol $\widetilde{D}_{\{b_n\}}$ of the operator $D_{\{b_n\}}$ has no radial limits anywhere on the unit circle \mathbb{T} .

Proof. Let $s_n := \sum_{k=1}^n k^{-1-ic}$. Since the series $\sum n^{-1-ic}$ is not convergent and $n^{-1-ic} = O(1/n)$, it follows from the Tauberian theorem that $\sum n^{-1-ic}$ is not Abel-convergent. On the other hand, in [8, p. 163] Hardy shows that $s_n + \frac{a_n}{ic}$ tends to a finite limit as *n* tends to infinite. Therefore it follows that the sequence $\{a_n\}$ cannot be Abel-convergent, for if it were we would get that $\{s_n\}$ is Abel-convergent, which is absurd. Obviously, $\{b_n\}$ is a bounded sequence, and therefore the diagonal operator $D_{\{b_n\}}$ is bounded on L_a^2 . Now by setting $T = D_{\{b_n\}}$ in the formula (1), we have

$$\widetilde{D}_{\{b_n\}}(\lambda) = \left(1 - |\lambda|^2\right)^2 \sum_{m=0}^{\infty} (m+1)b_m |\lambda|^{2m},$$

for every $\lambda \in \mathbb{D}$. Simple calculus shows that

$$\widetilde{D}_{\{b_n\}}(\lambda) = \left(1 - |\lambda|^2\right) \sum_{m=0}^{\infty} \left[(m+1)b_m - mb_{m-1} \right] |\lambda|^{2m},$$

that is

$$\widetilde{D}_{\{b_n\}}(\lambda) = \left(1 - |\lambda|^2\right) \sum_{m=0}^{\infty} a_m |\lambda|^{2m},$$
(2)

for all $\lambda \in \mathbb{D}$ (which shows that $\widetilde{D}_{\{b_n\}}$ is a radial function, that is $\widetilde{D}_{\{b_n\}}(\lambda) = \widetilde{D}_{\{b_n\}}(|\lambda|)$). Since $t := |\lambda|^2 < 1$ and $\{a_m\}$ is not Abel-convergent sequence, it follows easily from (2) that the Berezin symbol $\widetilde{D}_{\{b_n\}}$ of the diagonal operator $D_{\{b_n\}}$ has no radial limits anywhere on \mathbb{T} , which completes the proof of theorem. \Box

3. (e)-summability method

In this section we introduce a new summability method for sequences and series of complex numbers, which we call (e)-summability. We give in terms of Berezin symbols of an associate diagonal operator a criteria for this summability method, and prove regularity of this (e)-summability method (Theorem 3.2). Recall that a method is said to be regular if it sums every convergent sequence to its ordinary limit. It is well known for example that Cezaro, Abel and Borel methods are regular (see [8]).

Definition 3.1. Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a functional Hilbert space on some set Ω with reproducing kernel $k_{\lambda}(z) :=$ $\sum_{n\geq 0} \overline{e_n(\lambda)} e_n(z)$, where $e := \{e_n(z)\}_{n\geq 0}$ is an orthonormal basis of \mathcal{H} . Let $\{a_n\}$ be any sequence of complex numbers.

(1) We say that the sequence $\{a_n\}_{n \ge 0}$ is (e)-convergent to a if $\sum_{n=0}^{\infty} a_n |e_n(\lambda)|^2$ converges for all $\lambda \in \Omega$ and

$$\lim_{\lambda \to \zeta} \frac{1}{\sum_{n=0}^{\infty} |e_n(\lambda)|^2} \sum_{n=0}^{\infty} a_n |e_n(\lambda)|^2 = a,$$

for every $\zeta \in \partial \Omega$.

(2) We say that the series $\sum_{n=0}^{\infty} a_n$ is (e)-summable to a if $\sum_{n=0}^{\infty} a_n |e_n(\lambda)|^2$ converges for all $\lambda \in \Omega$ and $\lim_{\lambda \to \zeta} \sum_{n=0}^{\infty} a_n |e_n(\lambda)|^2 = a$ for every $\zeta \in \partial \Omega$.

It is easy to see that for $\mathcal{H} = H^2(\mathbb{D})$ (Hardy space) and $\mathcal{H} = L^2_a(\mathbb{C})$ (Fock space) our definitions of (e)-summability coincide with Abel and Borel summability, respectively.

For any bounded sequence $\{a_n\}_{n\geq 0}$ of complex numbers, let $D_{\{a_n\}}$ denote the diagonal operator on \mathcal{H} defined by $D_{\{a_n\}}e_n(z) = a_ne_n(z), n = 0, 1, 2, \dots$, with respect to the orthonormal basis $e = \{e_n(z)\}_{n \ge 0}$ of \mathcal{H} .

Following [11], we say that a functional Hilbert space ${\cal H}$ of complex-valued functions on some set ${\cal \Omega}$ is standard if the underlying set Ω is a subset of a topological space and the boundary $\partial \Omega$ is non-empty and has the property that $\{\frac{k_{\lambda}}{|k_{\lambda}|}\}$ converges weakly to 0 whenever λ tends to a point on the boundary.

The main result of this section is the following:

Theorem 3.2.

- (i) If $\{a_n\}_{n\geq 0}$ is a bounded sequence of complex numbers, then the sequence $\{a_n\}_{n\geq 0}$ (e)-converges to a if and only if $\lim_{\lambda \to \zeta} \widetilde{D}_{\{a_n\}}(\lambda) = a$, for every $\zeta \in \partial \Omega$.
- (ii) If $\{a_n\}_{n\geq 0}$ is a bounded sequence of complex numbers, then the series $\sum_{n=0}^{\infty} a_n$ is (e)-summable to a if and only if

$$\lim_{\lambda \to \zeta} \left(\sum_{n=0}^{\infty} \left| e_n(\lambda) \right|^2 \right) \widetilde{D}_{\{a_n\}}(\lambda) = a,$$

for every $\zeta \in \partial \Omega$.

(iii) If H is a standard functional Hilbert space, then (e)-summability method for sequences is regular.

Proof. Since $\{a_n\}_{n \ge 0}$ is a bounded sequence, $D_{\{a_n\}}$ is a bounded operator on \mathcal{H} . If \widehat{k}_{λ} is the normalized reproducing kernel of \mathcal{H} , then we have

$$\begin{split} \widetilde{D}_{\{a_n\}}(\lambda) &= \langle D_{\{a_n\}}\widehat{k}_{\lambda}, \widehat{k}_{\lambda} \rangle = \frac{1}{\|k_{\lambda}\|^2} \left\langle D_{\{a_n\}} \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z), k_{\lambda} \right\rangle \\ &= \frac{1}{\sum_{n=0}^{\infty} |e_n(\lambda)|^2} \left\langle \sum_{n=0}^{\infty} \overline{e_n(\lambda)} a_n e_n(z), k_{\lambda} \right\rangle = \frac{1}{\sum_{n=0}^{\infty} |e_n(\lambda)|^2} \sum_{n=0}^{\infty} a_n |e_n(\lambda)|^2, \end{split}$$

for all $\lambda \in \Omega$. Thus

$$\widetilde{D}_{\{a_n\}}(\lambda) = \frac{1}{\sum_{n=0}^{\infty} |e_n(\lambda)|^2} \sum_{n=0}^{\infty} a_n |e_n(\lambda)|^2, \quad \lambda \in \Omega.$$
(3)

Since $\sup_{\lambda \in \Omega} |\widetilde{D}_{[a_n]}(\lambda)| \leq \|\widetilde{D}_{[a_n]}\| = \sup_{n \geq 0} |a_n| < \infty$, formula (3) immediately implies the assertions (i) and (ii) of the theorem.

Now we prove the assertion (iii). Let $\{a_n\}_{n=0}^{\infty}$ converge to *a*. Then $D_{\{a_n-a\}}$ is a compact operator, and therefore $\widetilde{D}_{\{a_n-a\}}$ vanishes on the boundary $\partial \Omega$ (because \mathcal{H} is a standard functional Hilbert space), that is $\widetilde{D}_{\{a_n-a\}}(\lambda) \to 0$ as $\lambda \to \zeta \in \partial \Omega$. By considering this and formula (3), we have

$$\lim_{\lambda \to \zeta} \widetilde{D}_{\{a_n\}}(\lambda) = \lim_{\lambda \to \zeta} \frac{1}{\sum_{n=0}^{\infty} |e_n(\lambda)|^2} \sum_{n=0}^{\infty} a_n |e_n(\lambda)|^2 = \lim_{\lambda \to \zeta} \frac{1}{\sum_{n=0}^{\infty} |e_n(\lambda)|^2} \sum_{n=0}^{\infty} (a_n - a + a) |e_n(\lambda)|^2$$
$$= \lim_{\lambda \to \zeta} \frac{1}{\sum_{n=0}^{\infty} |e_n(\lambda)|^2} \sum_{n=0}^{\infty} (a_n - a) |e_n(\lambda)|^2 + a = \lim_{\lambda \to \zeta} \widetilde{D}_{\{a_n - a\}} + a,$$

which shows that (e)- $\lim_{n \to \infty} a_n = a$, as desired. The theorem is proved. \Box

The following corollaries can be obtained from Theorem 3.2 by setting $\mathcal{H} = H^2$:

Corollary 3.3. Let $\{a_n\}_{n \ge 0}$ be a bounded sequence of complex numbers, and let $D_{\{a_n\}}$ be the diagonal operator on H^2 with diagonal elements a_n , $n \ge 0$, with respect to the orthonormal basis $\{z^n\}_{n\ge 0}$ of H^2 . Then:

- (i) the sequence $\{a_n\}_{n\geq 0}$ is Abel convergent to a if and only if $\lim_{t\to 1^-} \widetilde{D}_{\{a_n\}}(t) = a$;
- (ii) the series $\sum_{n=0}^{\infty} a_n$ is Abel summable to a if and only if $\lim_{t\to 1^-} \frac{\tilde{D}_{(a_n)}(t)}{1-t} = a$.

Corollary 3.4. (See Abel theorem [8].) If $\{a_n\}_{n\geq 0}$ converges to a, then $\{a_n\}_{n\geq 0}$ is Abel convergent to a.

Corollary 3.5. (See Abel theorem [8].) If the series $\sum_{n=0}^{\infty} a_n$ converges to a, then $\sum_{n=0}^{\infty} a_n$ is Abel summable to a.

The next result gives in terms of Berezin symbols a formula for the sum of some convergent series of complex numbers. Its proof (which is omitted) uses Corollary 3.3 and L'Hôspital rule.

Corollary 3.6.

- (i) Let {a_n}_{n≥0} be any sequence of complex numbers. If the series ∑_{n=0}[∞] a_n converges to a, then a = lim_{t→1}- ^{D̃[a_n](t)}/_{1-t}, where D_{a_n} is the diagonal operator on H², with diagonal elements a_n, n ≥ 0, with respect to the orthonormal basis {zⁿ}_{n≥0}.
 (ii) Let {a_n}_{n≥0} be a sequence of complex numbers such that {na_n}_{n≥0} is Abel convergent to 0. If the series ∑_{n=0}[∞] a_n converges to a, then a = lim_{t→1}- ^{D̃[a_n](t)}/_{1-t}, where D_{{a_n}} is the diagonal operator on H², with diagonal elements a_n, n ≥ 0, with respect to the orthonormal basis {zⁿ}_{n≥0}.
- then $a = -\lim_{t \to 1^{-}} \widetilde{D}'_{\{a_n\}}(t)$.

Acknowledgements

The author is grateful to the referee for his useful remarks. This work is supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) with Project 109T590.

References

- [1] P. Ahern, M. Floers, W. Rudin, An invariant volume-mean-value property, J. Funct. Anal. 111 (1993) 233-254.
- [2] F.A. Berezin, Covariant and contravariant symbols for operators, Math. USSR-Izv. 6 (1972) 1117-1151.
- [3] F.A. Berezin, Quantization, Math. USSR-Izv. 8 (1974) 1109-1163.
- [4] C.A. Berger, L.A. Coburn, Toeplitz operators and quantum mechanics, J. Funct. Anal. 68 (1986) 273-299.
- [5] C.A. Berger, L.A. Coburn, Toeplitz operators on the Segal-Bergman space, Trans. Amer. Math. Soc. 301 (1987) 813-829.
- [6] M. Engliś, Functions invariant under the Berezin transform, J. Funct. Anal. 121 (1994) 223-254.
- [7] E. Fricain, Uniqueness theorems for analytic vector-valued functions, J. Math. Sci. (N. Y.) 101 (2000) 3193-3210; translation from Zap. Nauchn. Semin. POMI 247 (1997) 242-267.
- [8] G.H. Hardy, Divergent Series, Oxford, 1956.
- [9] H. Hedenmalm, B. Korenblum, K. Zhu, Theory of Bergman Spaces, Springer Verlag, 2000.
- [10] M.T. Karaev, On some problems related to Berezin symbols, C. R. Acad. Sci. Paris 340 (2005) 715-718.
- [11] E. Nordgren, P. Rosenthal, Boundary values of Berezin symbols, Oper. Theory Adv. Appl. 73 (1994) 362-368.
- [12] K. Zhu, Operator Theory in Function Spaces, Marcel Dekker, New York, 1990.
- [13] N. Zorboska, The Berezin transform and radial operators, Proc. Amer. Math. Soc. 131 (2003) 793-800.

1062