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## Mathematical Problems in Mechanics

# The obstacle problem for shallow shells in curvilinear coordinates

## Le problème d'obstacle en coordonnées curvilignes pour des coques peu profondes

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#### ABSTRACT

Starting from the 3D Signorini problem in presence of a plane obstacle, we justify the limit inequation of unilateral contact posed in a 2D domain. In particular, we show that we can uncouple the three covariant components of the limit Kirchhoff–Love displacement field so that the non-penetrability condition involves only the "transverse" component as this is the case in Cartesian framework.

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#### RÉSUMÉ

En partant du problème de Signorini en présence d'un obstacle plan on justifie l'inéquation limite du contact unilatéral posé dans un domaine 2D. On montre en particulier qu'on peut découpler les trois composantes covariantes du champ de déplacement limite de Kirchhoff–Love de telle sorte que la condition d'impénétrabilité ne porte que sur la composante « transverse », comme cela se passe dans le cas cartésien.

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#### Version française abrégée

L'objectif de cette Note est de justifier rigoureusement l'équation d'équilibre d'une coque peu profonde qui peut entrer en contact avec un obstacle plan. Ce problème a déjà été traité dans [4] dans le cadre cartésien. Nous l'abordons ici dans le cadre de la géométrie différentielle plus adaptée à l'étude des coques générales (cf. [1] pour l'équation d'équilibre dans le cas bilatéral).

Soit  $\varepsilon > 0$  un petit paramètre lié à l'épaisseur de la coque. La configuration de référence de la coque est l'adhérence du domaine  $\widehat{\Omega^{\varepsilon}}$ , elle est construite à partir d'un ouvert  $\omega$  de  $\mathbb{R}^2$ , d'une application  $\varphi^{\varepsilon} : \overline{\omega} \to \mathbb{R}^3$  de classe  $C^3(\overline{\omega})$  et d'un difféomorphisme  $\Phi^{\varepsilon}$  donné par (1) et tel que  $\widehat{\Omega^{\varepsilon}} = \Phi^{\varepsilon}(\Omega^{\varepsilon})$  où  $\Omega^{\varepsilon} = \omega \times ]-\varepsilon, \varepsilon[$ . Un système de coordonnées curvilignes est associé à  $\Omega^{\varepsilon}$ . On construit alors la base covariante  $g_i^{\varepsilon} = \partial_i \Phi^{\varepsilon}$  et la base contravariante  $g^{i,\varepsilon}$  associées à la coque, ce qui permet de repérer tout vecteur  $v^i g_i^{\varepsilon} = v_i g^{i,\varepsilon}$  par ses composantes covariantes  $(v_i)$  ou contravariantes  $(v^i)$ .

permet de repérer tout vecteur  $v^i \mathbf{g}_i^{\varepsilon} = v_i \mathbf{g}^{i,\varepsilon}$  par ses composantes covariantes  $(v_i)$  ou contravariantes  $(v^i)$ . On désigne par  $\Gamma_-^{\varepsilon} \cup \Gamma_+^{\varepsilon} \cup \Gamma_0^{\varepsilon}$  la frontière du cylindre  $\Omega^{\varepsilon}$  avec  $\Gamma_0^{\varepsilon} = \partial \omega \times [-\varepsilon, \varepsilon], \Gamma_+^{\varepsilon} = \omega \times \{\varepsilon\}, \Gamma_-^{\varepsilon} = \omega \times \{-\varepsilon\}$  et l'on suppose que la face "inférieure"  $\boldsymbol{\Phi}^{\varepsilon}(\Gamma_-^{\varepsilon})$  est susceptible d'entrer en contact (2) avec le plan horizontal situé à la cote  $-\varepsilon$ . Les conditions de contact unilatéral entre la coque et ce plan sont alors exprimées par les conditions (3), ce qui conduit classiquement au fait que l'équilibre est donné par l'unique solution  $\boldsymbol{u}^{\varepsilon} = (u_i^{\varepsilon})$  du problème (4) posé dans un cadre fonctionnel qui est le cône convexe (5).

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Une dilatation du domaine dans la direction normale à  $\omega$  accompagnée d'hypothèses sur l'application  $\boldsymbol{\varphi}^{\varepsilon}$  (6) et sur les forces ainsi qu'une mise à l'échelle des inconnues (7), justifiées par un développement asymptotique préalable, conduisent alors au problème d'équilibre (8) formulé dans le domaine fixe  $\Omega = \omega \times (1-1, 1)$  et qui possède une solution unique  $\mathbf{u}(\varepsilon) = (u_i(\varepsilon))$ . On note que la condition de non-pénétrabilité (9) couple les trois composantes covariantes des champs de déplacements admissibles.

Une analyse asymptotique ainsi que le Lemme 4.1, permettent (par l'introduction d'un nouveau champ de fonctions tests) d'établir que la solution  $u(\varepsilon)$  du problème de perturbations singulières (11) appartient à un cadre fonctionnel (12) où seule la composante covariante « normale » intervient (on se rapproche ainsi du cadre cartésien [4]).

Le Théorème 5.1 démontre l'existence et l'unicité d'un champ de déplacement limite sous la forme d'un champ de Kirchhoff-Love. Est ainsi justifié le problème d'obstacle limite bi-dimensionnel. La preuve complète de ce résultat de convergence sera donnée dans [5].

#### 1. Introduction

This Note aims at justifying rigorously the equilibrium equation of a shallow shell which may enter into contact with a plane obstacle. This justification has been given in [4] in the Cartesian framework. The main interest of this differential geometry approach based on the framework of curvilinear coordinates, as in [1], is that it is a first step towards the justification of the obstacle problem for general shells. The difficulty arises from the fact that the non-penetrability condition involves a coupling between the three covariant components of the displacement field which implies a specific treatment using Lemma 4.1. The justification of the model is given in Section 5 by a convergence result.

#### 2. The equilibrium problem

Let  $\omega$  be a bounded connected domain of  $\mathbb{R}^2$  with Lipschitz boundary and a set  $(x_1, x_2)$  of curvilinear coordinates. Let<sup>1</sup>  $\varphi^{\varepsilon} = (\varphi_i) : \overline{\omega} \to \mathbb{R}^3, \varphi^{\varepsilon} \in \mathbf{C}^3(\overline{\omega})$ , be an injective mapping whose third component  $\varphi_3$  is positive and depends upon a small parameter  $\varepsilon > 0$  (it has been shown in [2] that this dependence is needed to get the limit model of Novozilov's shallow shell). Then  $S^{\varepsilon} = \varphi^{\varepsilon}(\overline{\omega})$  is a surface embedded in  $\mathbb{R}^3$  which is the middle surface of the three-dimensional shell. We assume that the mapping  $\varphi^{\varepsilon}$  is such that the two vectors  $\mathbf{a}^{\varepsilon}_{\alpha}(x_1, x_2) = \frac{\partial \varphi^{\varepsilon}(x_1, x_2)}{\partial x_{\alpha}}$  are linearly independent, so that they form a basis of the tangent plane at each point of  $S^{\varepsilon}$ . The unit normal to  $S^{\varepsilon}$  is given by  $\boldsymbol{a}_{3}^{\varepsilon} = \boldsymbol{a}^{3,\varepsilon} = \frac{\boldsymbol{a}_{1}^{\varepsilon} \times \boldsymbol{a}_{2}^{\varepsilon}}{|\boldsymbol{a}_{1}^{\varepsilon} \times \boldsymbol{a}_{2}^{\varepsilon}|}$ , hence we have the covariant

basis  $(\boldsymbol{a}_{i}^{\varepsilon})$  attached to  $S^{\varepsilon}$  and the contravariant basis  $(\boldsymbol{a}^{i,\varepsilon})$  is then obtained by the relations  $\boldsymbol{a}_{i}^{\varepsilon} \cdot \boldsymbol{a}^{j,\varepsilon} = \delta_{i}^{j}$ .

The reference configuration of the shell is the three-dimensional domain  $\widehat{\Omega^{\varepsilon}} = \boldsymbol{\Phi}^{\varepsilon}(\Omega^{\varepsilon})$  classically built from  $\omega$  by the mapping  $\boldsymbol{\Phi}^{\varepsilon}: \overline{\Omega}^{\varepsilon} \to \mathbb{R}^2$  where  $\Omega^{\varepsilon}$  is the cylinder  $\Omega^{\varepsilon} = \omega \times (-\varepsilon, \varepsilon)$ :

$$\boldsymbol{\Phi}^{\varepsilon}(\boldsymbol{x}^{\varepsilon}) = \boldsymbol{\varphi}^{\varepsilon}(\boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{x}_3^{\varepsilon} \boldsymbol{a}_3^{\varepsilon}(\boldsymbol{x}_1, \boldsymbol{x}_2), \quad \boldsymbol{x}^{\varepsilon} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3^{\varepsilon}) \in \Omega^{\varepsilon}, \ (\boldsymbol{x}_1, \boldsymbol{x}_2) \in \boldsymbol{\omega}, \ \boldsymbol{x}_3^{\varepsilon} \in (-\varepsilon, \varepsilon).$$
(1)

Attached to the domain  $\widehat{\Omega^{\varepsilon}}$  let us define the three-dimensional covariant basis  $\mathbf{g}_{i}^{\varepsilon} = \partial_{i} \boldsymbol{\Phi}^{\varepsilon}$ ,  $(\mathbf{g}_{\alpha}^{\varepsilon} = \mathbf{a}_{\alpha}^{\varepsilon} + x_{3}^{\varepsilon} \partial_{\alpha} \mathbf{a}_{3}^{\varepsilon}, \mathbf{g}_{3}^{\varepsilon} = \mathbf{a}_{3}^{\varepsilon})$ and the contravariant basis  $\mathbf{g}^{i,\varepsilon}$ . Therefore any three-dimensional vector field attached to the shell reads  $v^i \mathbf{g}^{\varepsilon}_i$  or  $v_i \mathbf{g}^{i,\varepsilon}$ . Let  $\Gamma^{\varepsilon}_{-} \cup \Gamma^{\varepsilon}_{+} \cup \Gamma^{\varepsilon}_{0}$  be the boundary of the cylinder  $\Omega^{\varepsilon}$  with  $\Gamma^{\varepsilon}_{+} = \omega \times \{\varepsilon\}$ ,  $\Gamma^{\varepsilon}_{-} = \omega \times \{-\varepsilon\}$ ,  $\Gamma^{\varepsilon}_{0} = \partial\omega \times [-\varepsilon, \varepsilon]$ . Assume that the "upper" boundary  $\boldsymbol{\Phi}^{\varepsilon}(\Gamma^{\varepsilon}_{+})$  is stress free, the "lateral" boundary  $\boldsymbol{\Phi}^{\varepsilon}(\Gamma^{\varepsilon}_{0})$  is clamped, and the "lower" boundary  $\boldsymbol{\Phi}^{\varepsilon}(\Gamma^{\varepsilon}_{-})$ may enter into contact with a rigid plane obstacle. It has been observed in [4] that it is convenient and not restrictive to assume that the contact holds with a horizontal plane at the level  $-\varepsilon$ . Then the natural form of the non-penetrability of the reference configuration of the shell into the rigid obstacle reads<sup>2</sup>:

$$v_i \boldsymbol{g}^{i,\varepsilon} \big|_3 \ge -\boldsymbol{\Phi}^{\varepsilon} \big|_3 - \varepsilon \quad \text{on } \Gamma^{\varepsilon}_{-}.$$
<sup>(2)</sup>

It is known that inequality (2) must be supplied with two additional conditions:

- (i) If a point of the corresponding part of the boundary is not in contact, i.e., if the inequality is strict in (2), then there is no reaction of the obstacle on the 3D body at this point;
- (ii) A non-zero reaction of the obstacle on the body is possible only at a point of the boundary which is in contact, i.e., where the equality holds in (2).

Then, the whole set of unilateral contact conditions on  $\Gamma_{-}^{\varepsilon} = \omega \times \{-\varepsilon\}$  is:

Latin exponents and indices take their values in the set {1; 2; 3}, Greek exponents and indices (except  $\varepsilon$ ) take their values in the set {1; 2}, Einstein convention for repeated exponents and indices is used and bold face letters represent vectors or vector spaces.

<sup>&</sup>lt;sup>2</sup> The notation  $\cdot|_3$  represents the third Cartesian component of a vector or of the mapping  $\boldsymbol{\Phi}^{\varepsilon}$ .

$$\begin{cases} v_i \boldsymbol{g}^{i,\varepsilon}|_3 \ge -\boldsymbol{\Phi}^{\varepsilon}|_3 - \varepsilon, \\ -\left[ (\boldsymbol{\sigma}^{\varepsilon} \boldsymbol{a}_3^{\varepsilon})^i \boldsymbol{g}_i^{\varepsilon} \right]|_3 \ge 0, \\ (v_i \boldsymbol{g}^{i,\varepsilon}|_3 + \boldsymbol{\Phi}^{\varepsilon}|_3 + \varepsilon) \left[ (\boldsymbol{\sigma}^{\varepsilon} \boldsymbol{a}_3^{\varepsilon})^i \boldsymbol{g}_i^{\varepsilon} \right]|_3 = 0. \end{cases}$$
(3)

When subjected to volume forces  $f^{\varepsilon}$ , the elastic body undergoes a displacement field  $u_i^{\varepsilon} g^{i,\varepsilon}$  which is the *unique solution* of the following variational problem:

$$\mathcal{P}^{\varepsilon}(\boldsymbol{u}^{\varepsilon}) = \begin{cases} \text{Find } \boldsymbol{u}^{\varepsilon} = (u_{i}^{\varepsilon}) \in \boldsymbol{K}^{\varepsilon}(\Omega^{\varepsilon}) \text{ such that for all } \boldsymbol{v} = (v_{i}) \in \boldsymbol{K}^{\varepsilon}(\Omega^{\varepsilon}) \\ \int_{\Omega^{\varepsilon}} g^{ijkl,\varepsilon} e_{k\parallel l}^{\varepsilon} (\boldsymbol{u}^{\varepsilon}) e_{i\parallel j}^{\varepsilon} (\boldsymbol{v} - \boldsymbol{u}^{\varepsilon}) \sqrt{g^{\varepsilon}} \, \mathrm{d} x^{\varepsilon} \geqslant \int_{\Omega^{\varepsilon}} f^{i,\varepsilon} (v_{i} - u_{i}^{\varepsilon}) \sqrt{g^{\varepsilon}} \, \mathrm{d} x^{\varepsilon} \end{cases}$$
(4)

where the strain tensor is defined as  $e_{i||j}^{\varepsilon}(\mathbf{v}) = \frac{1}{2}(\frac{\partial v_j}{\partial x_i^{\varepsilon}} + \frac{\partial v_i}{\partial x_j^{\varepsilon}}) - \mathcal{G}_{ij}^{p,\varepsilon} v_p$ , with  $\mathcal{G}_{ij}^{k,\varepsilon}$  the Christoffel's symbols and  $g^{\varepsilon} = \det(g_{ij,\varepsilon})$  with  $g_{ij,\varepsilon} = \mathbf{g}_i^{\varepsilon} \cdot \mathbf{g}_j^{\varepsilon}$  the components of the metric tensor. Using conditions (3), the functional framework  $\mathbf{K}^{\varepsilon}(\Omega^{\varepsilon})$  is the convex cone given by:

$$\boldsymbol{K}^{\varepsilon}(\boldsymbol{\Omega}^{\varepsilon}) = \left\{ \boldsymbol{\nu} \in \boldsymbol{H}^{1}(\boldsymbol{\Omega}^{\varepsilon}), \ \boldsymbol{\nu} = \boldsymbol{0} \text{ on } \boldsymbol{\Gamma}_{0}^{\varepsilon}, \ \boldsymbol{\nu}_{i} \boldsymbol{g}^{i,\varepsilon} \right\}_{3} \ge -\boldsymbol{\Phi}^{\varepsilon} |_{3} - \varepsilon \text{ on } \boldsymbol{\Gamma}_{-}^{\varepsilon} \right\}.$$
(5)

In the case of a linear isotropic material with Lamé coefficients  $\lambda$ ,  $\mu$ , the fourth order elasticity tensor reads  $g^{ijkl,\varepsilon} = \lambda g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu (g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon})$  and  $(g^{ij,\varepsilon})$  are the contravariant components of the metric tensor.

**Remark 1.** The natural setting of the equilibrium problem is the reference configuration  $\widehat{\Omega^{\varepsilon}} = \Phi^{\varepsilon}(\Omega^{\varepsilon})$ . To obtain formulations (2), (3) and (4) in  $\Omega^{\varepsilon}$  we recall that the mapping  $\Phi^{\varepsilon}$  is a  $C^1$ -diffeomorphism for  $\varphi^{\varepsilon} \in C^3(\overline{\omega})$  and  $\varepsilon$  small enough.

#### 3. Formulation in the fixed domain ${\boldsymbol \varOmega}$

Now we transform the domain  $\Omega^{\varepsilon} = \omega \times (-\varepsilon, \varepsilon)$  of thickness  $2\varepsilon$  into the fixed cylindrical domain  $\Omega = \omega \times (-1, 1)$  independent of  $\varepsilon$ , with  $\partial \Omega = \Gamma_{-} \cup \Gamma_{+} \cup \Gamma_{0}$ , via the simple dilatation:

$$\Omega^{\varepsilon} \ni x^{\varepsilon} = \left(x_1, x_2, x_3^{\varepsilon}\right) \to x = (x_1, x_2, x_3) \in \Omega, \quad x_3 = \frac{1}{\varepsilon} x_3^{\varepsilon}.$$

In order to get a limit model of problem  $\mathcal{P}^{\varepsilon}(\boldsymbol{u}^{\varepsilon})$  when the thickness  $2\varepsilon$  goes to zero, we make, as in [2], appropriate assumptions on the data and scalings of the unknowns. More precisely:

• We assume that the Lamé coefficients  $\lambda, \mu$  are independent of  $\varepsilon$  and the mapping  $\varphi^{\varepsilon} : \overline{\omega} \to \mathbb{R}^3$  which defines the middle surface  $S^{\varepsilon}$  of the shell reads

$$\boldsymbol{\varphi}^{\varepsilon} = (\varphi_1, \varphi_2, \varepsilon \varphi_3), \quad \text{with } \varphi_i \text{ independent of } \varepsilon \text{ and } \varphi_3 \ge 0.$$
 (6)

• With each  $v(x^{\varepsilon})$  defined in  $\Omega^{\varepsilon}$ , we associate new functions  $v(\varepsilon)(x)$  defined in  $\Omega$  by  $v(x^{\varepsilon}) = v(\varepsilon)(x)$ . For example, for the covariant basis  $a_i^{\varepsilon}(x^{\varepsilon}) = a_i(\varepsilon)(x)$  we have

$$\boldsymbol{a}_{\alpha}(\varepsilon) = \partial_{\alpha}\boldsymbol{\varphi}(\varepsilon) = \begin{pmatrix} \partial_{\alpha}\varphi_1 \\ \partial_{\alpha}\varphi_2 \\ \varepsilon \partial_{\alpha}\varphi_3 \end{pmatrix}, \quad \boldsymbol{a}_{3}(\varepsilon) = \frac{1}{\sqrt{d(\varepsilon)}} \begin{pmatrix} \varepsilon(\partial_{1}\varphi_{2}\partial_{2}\varphi_{3} - \partial_{2}\varphi_{2}\partial_{1}\varphi_{3}) \\ \varepsilon(\partial_{1}\varphi_{3}\partial_{2}\varphi_{1} - \partial_{1}\varphi_{1}\partial_{2}\varphi_{3}) \\ \partial_{1}\varphi_{1}\partial_{2}\varphi_{2} - \partial_{1}\varphi_{2}\partial_{2}\varphi_{1} \end{pmatrix},$$

with  $d(\varepsilon) = \varepsilon^2 (\partial_1 \varphi_2 \partial_2 \varphi_3 - \partial_2 \varphi_2 \partial_1 \varphi_3)^2 + \varepsilon^2 (\partial_1 \varphi_3 \partial_2 \varphi_1 - \partial_1 \varphi_1 \partial_2 \varphi_3)^2 + (\partial_1 \varphi_1 \partial_2 \varphi_2 - \partial_1 \varphi_2 \partial_2 \varphi_1)^2.$ 

- We assume that the volume force is of the form  $\varepsilon^2 f^{\alpha} \mathbf{g}_{\alpha}(\varepsilon) + \varepsilon^3 f^3 \mathbf{g}_3(\varepsilon)$ .
- Finally, as in [3,1] the displacement field  $u(\varepsilon)$  and the test-functions v are rescaled:

$$u_{\alpha}^{\varepsilon}(x^{\varepsilon}) = \varepsilon^{2} u_{\alpha}(\varepsilon)(x), \qquad u_{3}^{\varepsilon}(x^{\varepsilon}) = \varepsilon u_{3}(\varepsilon)(x), \qquad v_{\alpha}(x^{\varepsilon}) = \varepsilon^{2} v_{\alpha}(x), \qquad v_{3}(x^{\varepsilon}) = \varepsilon v_{3}(x).$$
(7)

Hence the displacement field now reads  $\varepsilon^2 u_{\alpha}(\varepsilon) \mathbf{g}^{\alpha}(\varepsilon) + \varepsilon u_3(\varepsilon) \mathbf{g}^3(\varepsilon)$ . Let us now focus on the contact condition; after scaling its expression on  $\Gamma_-$  reads:  $\varepsilon^2 v_{\alpha} \mathbf{g}^{\alpha}(\varepsilon)|_3 + \varepsilon v_3 \mathbf{g}^3(\varepsilon)|_3 \ge -\varepsilon \varphi_3 + \varepsilon \mathbf{a}_3(\varepsilon)|_3 - \varepsilon$  on  $\Gamma_-$ . Then the equilibrium problem in curvilinear coordinates reads

$$\widetilde{\mathcal{P}}(\varepsilon)(\boldsymbol{u}(\varepsilon)) \quad \begin{cases} \operatorname{Find} \boldsymbol{u}(\varepsilon) \in \widetilde{\boldsymbol{K}}(\varepsilon)(\Omega) \text{ such that for all } \boldsymbol{v} \in \widetilde{\boldsymbol{K}}(\varepsilon)(\Omega), \\ \int_{\Omega} g^{ijkl}(\varepsilon) e_{k||l}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{i||j}(\varepsilon)(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) \sqrt{g(\varepsilon)} \, \mathrm{d}x \geqslant \int_{\Omega} f^{i}(\varepsilon)(v_{i} - u_{i}(\varepsilon)) \sqrt{g(\varepsilon)} \, \mathrm{d}x \end{cases}$$
(8)

where  $g^{ijkl}(\varepsilon) = \lambda g^{ij}(\varepsilon)g^{kl}(\varepsilon) + \mu(g^{ik}(\varepsilon)g^{jl}(\varepsilon) + g^{il}(\varepsilon)g^{jk}(\varepsilon))$  are the scaled contravariant coefficients of elasticity,  $g_{ij}(\varepsilon) = \beta g^{ijkl}(\varepsilon)g^{kl}(\varepsilon)g^{kl}(\varepsilon)$  $\mathbf{g}_i(\varepsilon) \cdot \mathbf{g}_i(\varepsilon)$  and  $g^{ij}(\varepsilon) = \mathbf{g}^i(\varepsilon) \cdot \mathbf{g}^j(\varepsilon)$  are respectively the covariant and contravariant components of the metric tensor, and  $g(\varepsilon) = \det(g_{ij}(\varepsilon))$ . The functions  $e_{i\parallel i}(\varepsilon)(\mathbf{v})$  are the scaled covariant components of the linearized strain tensor. The set of admissible displacement fields is now

$$\widetilde{\boldsymbol{K}}(\varepsilon)(\Omega) = \left\{ \boldsymbol{\nu} \in \boldsymbol{H}^{1}(\Omega), \ \boldsymbol{\nu} = \boldsymbol{0} \text{ on } \Gamma_{0}, \ \varepsilon \nu_{\alpha} \boldsymbol{g}^{\alpha}(\varepsilon) \big|_{3} + \nu_{3} \boldsymbol{a}^{3}(\varepsilon) \big|_{3} \ge -\varphi_{3} + \left( \boldsymbol{a}_{3}(\varepsilon) \big|_{3} - 1 \right) \text{ on } \Gamma_{-} \right\}.$$
(9)

We observe that the tangential and the normal components of the displacement fields are coupled by the non-penetrability condition (9) while in the Cartesian framework the non-penetrability condition was a constraint only on the normal component of the displacement field.

#### 4. Asymptotic analysis

The asymptotic analysis consists in three steps:

(i) **Taylor expansion of the data around the middle surface.** For example for the covariant basis:

$$\boldsymbol{a}_{\alpha}(\varepsilon) = \begin{pmatrix} \partial_{\alpha}\varphi_1 \\ \partial_{\alpha}\varphi_2 \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ 0 \\ \partial_{\alpha}\varphi_3 \end{pmatrix} + \cdots, \qquad \boldsymbol{a}_3 = \boldsymbol{a}^3(\varepsilon) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \varepsilon \frac{1}{a} \begin{pmatrix} \partial_1\varphi_1\partial_2\varphi_3 - \partial_2\varphi_1\partial_1\varphi_3 \\ \partial_1\varphi_3\partial_2\varphi_1 - \partial_1\varphi_1\partial_2\varphi_3 \\ 0 \end{pmatrix} + \cdots,$$

with  $a = \partial_1 \varphi_1 \partial_2 \varphi_2 - \partial_1 \varphi_2 \partial_2 \varphi_1$ .

(ii) Influence of the scalings on the unknowns. Step (i) gives the leading terms of the strain tensor:

$$e_{\alpha\parallel\beta}(\varepsilon)(\mathbf{v}) = \varepsilon^{2} \left( e_{\alpha\parallel\beta}^{\varphi}(\mathbf{v}) + \varepsilon e_{\alpha\beta}^{\sharp}(\varepsilon,\varphi;\mathbf{v}) \right), \qquad e_{\alpha\parallel3}(\varepsilon)(\mathbf{v}) = \varepsilon \left( e_{\alpha\parallel3}^{\varphi}(\mathbf{v}) + \varepsilon e_{\alpha3}^{\sharp}(\varepsilon,\varphi;\mathbf{v}) \right)$$
$$e_{3\parallel3}(\varepsilon)(\mathbf{v}) = e_{3\parallel3}^{\varphi}(\mathbf{v})$$

with  $e^{\varphi}$  independent of  $\varepsilon$  ( $\Gamma^{\sigma}_{\alpha\beta}$  and  $b_{\alpha\beta}$  are the limits of Christoffel symbols  $\mathcal{G}^{\sigma}_{\alpha\beta}(\varepsilon)$  and  $\mathcal{G}^{3}_{\alpha\beta}(\varepsilon)$ ).

$$e^{\varphi}_{\alpha\parallel\beta}(\mathbf{v}) = \frac{1}{2}(\partial_{\alpha}v_{\beta} + \partial_{\beta}v_{\alpha}) - \Gamma^{\sigma}_{\alpha\beta}v_{\sigma} - b_{\alpha\beta}v_{3}, \qquad e^{\varphi}_{\alpha\parallel3}(\mathbf{v}) = \frac{1}{2}(\partial_{\alpha}v_{3} + \partial_{3}v_{\alpha}), \qquad e^{\varphi}_{3\parallel3}(\mathbf{v}) = \partial_{3}v_{3}. \tag{10}$$

(iii) Decoupling of the in-plane and transverse covariant components of the displacement fields. This is done by the following lemma:

**Lemma 4.1.** For  $\varepsilon > 0$ , we consider the test function  $w_3(\varepsilon) = \varepsilon v_\alpha g^\alpha(\varepsilon)|_3 + v_3 g^3(\varepsilon)|_3$  with  $v_i \in H^1(\Omega)$ .

- (i) The non-penetrability condition  $\varepsilon v_{\alpha} \mathbf{g}^{\alpha}(\varepsilon)|_{3} + v_{3} \mathbf{g}^{3}(\varepsilon)|_{3} \ge -\varphi_{3} + \mathbf{a}_{3}(\varepsilon)|_{3} 1$  now reads  $w(\varepsilon) \ge -\varphi_{3} + \mathbf{a}_{3}(\varepsilon)|_{3} 1$  on  $\Gamma_{-}$ . (ii) Let  $z \in L^{2}(\Omega)$ , then  $\int_{\Omega} z\partial_{i}v_{3} dx = \int_{\Omega} z\partial_{i}w_{3}(\varepsilon) dx \varepsilon^{2} \int_{\Omega} z \mathbf{g}_{i}^{\sharp}(\varepsilon, \mathbf{w}(\varepsilon), \boldsymbol{\varphi}) dx$  (without summation), where the remainders  $\mathbf{g}_{i}^{\sharp}(\varepsilon, \mathbf{w}(\varepsilon), \boldsymbol{\varphi}) dx$  (without summation), where the remainders  $\mathbf{g}_{i}^{\sharp}(\varepsilon, \mathbf{w}(\varepsilon), \boldsymbol{\varphi}) dx$  (without summation). are bounded in  $L^{2}(\Omega)$ .

The new formulation of the equilibrium problem is thus:

$$\mathcal{P}(\varepsilon)(\mathbf{u}(\varepsilon)) \begin{cases} \text{Find } \mathbf{u}(\varepsilon) \in \mathbf{K}(\varepsilon)(\Omega) \text{ such that for all } \mathbf{v} \in \mathbf{K}(\varepsilon)(\Omega), \\ \iint_{\Omega} \left[ a^{\alpha\beta\sigma\tau} e^{\varphi}_{\alpha\parallel\beta}(\mathbf{u}(\varepsilon)) e^{\varphi}_{\sigma\parallel\tau}(\mathbf{v} - \mathbf{u}(\varepsilon)) + \frac{1}{\varepsilon^{2}} a^{\alpha\beta33} e^{\varphi}_{\alpha\parallel\beta}(\mathbf{u}(\varepsilon)) e^{\varphi}_{3\parallel3}(\mathbf{v} - \mathbf{u}(\varepsilon)) \right. \\ \left. + \frac{1}{\varepsilon^{2}} a^{\alpha\beta33} e^{\varphi}_{33}(\mathbf{u}(\varepsilon)) e^{\varphi}_{\alpha\parallel\beta}(\mathbf{v} - \mathbf{u}(\varepsilon)) + \frac{4}{\varepsilon^{2}} a^{\alpha3\beta3} e^{\varphi}_{\alpha\parallel3}(\mathbf{u}(\varepsilon)) e^{\varphi}_{\beta\parallel3}(\mathbf{v} - \mathbf{u}(\varepsilon)) \right. \\ \left. + \frac{1}{\varepsilon^{4}} a^{3333} e^{\varphi}_{33}(\mathbf{u}(\varepsilon)) e^{\varphi}_{3\parallel3}(\mathbf{v} - \mathbf{u}(\varepsilon)) \right] \sqrt{a} \, dx + \varepsilon \int_{\Omega} B^{\sharp}(\varepsilon, \varphi, \mathbf{u}(\varepsilon), \mathbf{v}) \, dx \\ \left. \geq \int_{\Omega} f^{i}(\mathbf{v}_{i} - \mathbf{u}_{i}(\varepsilon)) \sqrt{a} \, dx + \varepsilon \int_{\Omega} L^{\sharp}(\varepsilon, \varphi, \mathbf{u}(\varepsilon), \mathbf{v}) \, dx, \end{cases}$$

$$(11)$$

where the remainders  $B^{\sharp}$  and  $L^{\sharp}$  are bounded in  $L^{2}(\Omega)$  when  $\boldsymbol{u}(\varepsilon)$  and  $\boldsymbol{v}$  are bounded in  $H^{1}(\Omega)$ . The functional framework is:

$$\boldsymbol{K}(\varepsilon)(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{H}^{1}(\Omega), \, \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_{0}, \, v_{3} \ge -\varphi_{3} + \boldsymbol{a}_{3}(\varepsilon) \big|_{3} - 1 \text{ on } \Gamma_{-} \right\}.$$

$$(12)$$

We are now in position to give the main result of this Note.

#### 5. The limit problem

**Theorem 5.1.** Assume  $\mathbf{f} \in \mathbf{L}^2(\Omega), \boldsymbol{\varphi} \in \mathbf{C}^3(\overline{\omega})$ . Then

- (i) The family  $\{\mathbf{u}(\varepsilon)\}_{\varepsilon>0}$  solution to (11)-(12) converges strongly in the cone  $\mathbf{K}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega), \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0, \mathbf{v}_3 \ge -\varphi_3 \text{ on } \Gamma_-\}$ to a Kirchhoff-Love displacement field  $u_\alpha = \zeta_\alpha - x_3 \partial_\alpha \zeta_3, u_3 = \zeta_3$ ;
- (ii) The field  $\boldsymbol{\zeta} = (\zeta_i) \in V(\omega) \times V(\omega) \times K_3(\omega)$  with

$$V(\omega) = \left\{ \eta_{\alpha} \in H^{1}(\omega), \ \eta_{\alpha} = 0 \text{ on } \partial \omega \right\}, \qquad K_{3}(\omega) = \left\{ \eta_{3} \in H^{2}(\omega), \ \eta_{3} = \partial_{\nu}\eta_{3} = 0 \text{ on } \partial \omega, \ \eta_{3} \ge -\varphi_{3} \text{ in } \omega \right\}$$

is the unique solution to the following limit contact problem:

$$\int_{\omega} b^{\alpha\beta\gamma\tau} \left( \frac{2}{3} \left( \partial_{\alpha\beta}\zeta_{3} - \Gamma_{\alpha\beta}^{\kappa} \partial_{\kappa}\zeta_{3} \right) \left( \partial_{\sigma\tau} (\eta_{3} - \zeta_{3}) - \Gamma_{\sigma\tau}^{\kappa} \partial_{\kappa} (\eta_{3} - \zeta_{3}) \right) + 2e_{\alpha\parallel\beta}^{\varphi}(\zeta) e_{\sigma\parallel\tau}^{\varphi}(\eta - \zeta) \right) \sqrt{a} \, \mathrm{d}x_{1} \, \mathrm{d}x_{2}$$

$$\geqslant \int_{\omega} \left( p^{i}(\eta_{i} - \zeta_{i}) + \int_{\omega} s^{\alpha} \partial_{\alpha}(\eta_{3} - \zeta_{3}) \right) \sqrt{a} \, \mathrm{d}x_{1} \, \mathrm{d}x_{2}, \quad \forall \eta \in V(\omega) \times V(\omega) \times K_{3}(\omega),$$

with  $e_{\alpha \parallel \beta}^{\varphi}$  given in (10),

$$p^{i} = \int_{-1}^{1} f^{i} dx_{3}, \qquad s^{\alpha} = \int_{-1}^{1} x_{3} f^{\alpha} dx_{3}, \qquad b^{\alpha\beta\gamma\delta} = \frac{2\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\gamma\delta} + \mu \left(a^{\alpha\gamma} a^{\beta\delta} + a^{\alpha\delta} a^{\beta\gamma}\right)$$

and  $a^{\alpha\beta}$  are the contravariant components of the limit metric tensor  $g^{\alpha\beta}(\varepsilon)$ .

The main points of the proof (which is given in full in [5]) are the use of Lemma 4.1, the inclusion  $K(\Omega) \subset K(\varepsilon)(\Omega)$ and the fact that the map

$$\left\{\sum_{i,j} \left| e^{\varphi}_{i\parallel j}(\mathbf{v}) \right|^2_{0,\mathcal{Q}} \right\}^{1/2}$$

defines a norm on the cone  $K(\varepsilon)(\Omega)$  which is equivalent to the norm induced by  $H^1(\Omega)$ , and the map

$$\left\{\sum_{\alpha,\beta}\left|e_{\alpha\parallel\beta}^{\varphi}(\eta)\right|_{0,\omega}^{2}\right\}^{1/2}+\left\{\sum_{\alpha,\beta}\left|\partial_{\alpha\beta}\eta_{3}-\Gamma_{\alpha\beta}^{\kappa}\partial_{\kappa}\eta_{3}\right|_{0,\omega}^{2}\right\}^{1/2}$$

defines a norm on the set  $V(\omega) \times V(\omega) \times K_3(\omega)$  which is equivalent to the norm induced by  $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ . As a conclusion let us remark that

- The mapping  $\varphi^{\varepsilon}$  considered in this Note has a more general form than  $\varphi^{\varepsilon} = (x_1, x_2, \varphi_3^{\varepsilon})$  considered in [1].
- When the convex cone of contact is replaced by the vector space  $K_3(\omega) = \{\eta_3 \in H^2(\omega), \eta_3 = \partial_{\nu}\eta_3 = 0 \text{ on } \partial\omega\}$  (i.e., in the bilateral case), we get the general form of the virtual work principle for a shallow shell in curvilinear coordinates.
- Without any restriction we could as well assume that the shell is clamped only on a part  $\gamma_0 \times [-\varepsilon, \varepsilon]$  of the lateral boundary,  $\gamma_0 \subset \partial \omega$ , length  $\gamma_0 > 0$  and that surface forces are applied on the "upper" surface  $\boldsymbol{\Phi}^{\varepsilon}(\Gamma_{\pm}^{\varepsilon})$ .

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