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# Number Theory

## *m*-Bigness in compatible systems

### m-Bigness dans les systèmes compatibles

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#### ABSTRACT

Taylor–Wiles type lifting theorems allow one to deduce that if  $\rho$  is a "sufficiently nice" *l*-adic representation of the absolute Galois group of a number field whose semi-simplified reduction modulo *l*, denoted  $\overline{\rho}$ , comes from an automorphic representation then so does  $\rho$ . The recent lifting theorems of Barnet-Lamb–Gee–Geraghty–Taylor impose a technical condition, called *m*-big, upon the residual representation  $\overline{\rho}$ . Snowden–Wiles proved that for a sufficiently irreducible compatible system of Galois representations, the residual images are *big* at a set of places of Dirichlet density 1. We demonstrate the analogous result in the *m*-big setting using a mild generalization of their argument.

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#### RÉSUMÉ

Les théorèmes de type Taylor–Wiles indiquent qu'une représentation *l*-adique du groupe Galois d'un corps de nombre est automorphe si sa réduction modulo *l* est automorphe et si cette représentation satisfait de bonnes propriétés. Une condition technique mais cruciale qui apparaît dans le travail récent de Barnet-Lamb–Gee–Geraghty–Taylor est que la représentation résiduelle soit *m-big*. Snowden–Wiles ont demontré que pour un système compatible de représentations suffisamment irréductibles, les images résiduelles sont alors *big* pour un ensemble de Dirichlet densité 1. Nous démonstration de Snowden–Wiles.

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#### Version française abrégée

Le but de cette Note est de faire les modifications nécessaires au travail de Snowden–Wiles [3] afin de généraliser leurs résultats sur 1-*big* à *m*-*big*. La définition de *m*-*big* est rappelée dans Definition 1.1. Elle est une condition technique qui apparaît dans les généralisations récentes de la méthode de Taylor–Wiles aux groupes unitaires (cf. [1]). Le résultat principal de cet article est le théorème suivant (cf. Theorem 1.3) :

**Théorème 0.1.** Soient  $m \in \mathbf{N}$ , F un corps de nombres, E une extension galoisienne de  $\mathbf{Q}$  et L un ensemble plein de places de E. Pour tous  $w \in L$ , soient  $\rho_w : \operatorname{Gal}(\overline{\mathbf{Q}}/F) \to \operatorname{GL}_n(E_w)$  une représentation continue semi-simple et  $\Delta_w \subset \operatorname{Gal}(\overline{\mathbf{Q}}/F)$  un sous-groupe normal ouvert. Supposons que les propriétés suivantes sont satisfaites :

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- Les  $\rho_w$  forment un système compatible de représentations.
- Pour tout  $w \in L$ , la restriction de  $\rho_w$  à n'importe quel sous-groupe ouvert de de Gal( $\overline{\mathbf{Q}}/F$ ) est absolument irréductible.
- Pour tout  $w \in L$ ,  $Gal(\overline{\mathbf{Q}}/F)/\Delta_w$  est cyclique d'ordre premier à la caractéristique résiduelle de w.
- $[\operatorname{Gal}(\overline{\mathbf{Q}}/F):\Delta_w] \to \infty$  lorsque  $w \to \infty$ .

Alors il existe un ensemble de places P de **Q** de densité  $1/[E:\mathbf{Q}]$ , qui sont toutes totalement déployées dans E, et telles que, pour tout  $w \in L$  au-dessus une place  $l \in P$ :

- (i)  $\overline{\rho}_{w}(\Delta_{w})$  est un sous-groupe m-big de  $GL_{n}(\mathbf{F}_{l})$ .
- (ii) [ker ad  $\overline{\rho}_w : \Delta_w \cap \ker \operatorname{ad} \overline{\rho}_w] > m$ .

Remarque 0.1. La première partie de ce théorème est une généralisation du résultat principal de Snowden-Wiles [3]. La démonstration suit leurs arguments, en appliquant Proposition 3.1 au lieu de [3, Proposition 4.1].

La deuxième partie est un résultat de Barnet-Lamb-Gee-Geraghty-Taylor qui est apparu à l'origine dans [1].

Le plan de cet article est pareil à celui de [3]. Section 2 démontre quelques propriétés de *m*-big qui étaient démontrées pour 1-big dans [3, §2]. Section 3 démontre Proposition 3.1 qui améliore légèrement [3, Proposition 4.1]. Ce résultat est appliqué dans Section 4 pour démontrer que l'image de certaines représentations algébriques est m-big (cf. Proposition 4.1 qui améliore [3, Proposition 5.1]). Finalement, Sections 5 et 6 appliquent ce résultat pour démontrer le théorème principal.

#### 1. Introduction

We begin by recalling the condition m-big (cf. [2, Definition 7.2]). Let m be a positive integer, let l be a rational prime, let k be a finite field of characteristic l, let V be a finite-dimensional k-vector space and let  $G \subset GL(V)$  be a subgroup. For  $g \in GL(V)$  and  $\alpha \in k$ , we shall write  $h_g$  for the *characteristic polynomial* of g and  $V_{g,\alpha}$  for the  $\alpha$ -generalized eigenspace of g.

**Definition 1.1.** The subgroup G is said to be *m*-big if it satisfies the following properties:

- (B1) The group *G* has no non-trivial quotient of *l*-power order.
- (B2) The space V is absolutely irreducible as a G-module.
- (B3)  $H^1(G, \mathrm{ad}^\circ V) = 0.$
- (B4) For all irreducible *G*-submodules *W* of ad *V*, there exists  $g \in G$ ,  $\alpha \in k$  and  $f \in W$  such that:
  - The composite  $V_{g,\alpha} \hookrightarrow V \xrightarrow{f} V \twoheadrightarrow V_{g,\alpha}$  is non-zero.  $\alpha$  is a simple root of  $h_g$ .

  - If  $\beta \in \overline{k}$  is another root of  $h_g$  then  $\alpha^m \neq \beta^m$ .

**Remark 1.2.** The condition *big* appearing in [3] corresponds here to the condition 1-*big*.

Our main result is the following:

**Theorem 1.3.** Let F be a number field, let E be a Galois extension of Q, let L be a full set of places of E and for each  $w \in L$ , let  $\rho_w$ : Gal( $\bar{\mathbf{Q}}/F$ )  $\rightarrow$  GL<sub>n</sub>( $E_w$ ) be a continuous representation and let  $\Delta_w \subset$  Gal( $\bar{\mathbf{Q}}/F$ ) be a normal open subgroup. Assume that the following properties are satisfied:

- The  $\rho_w$  form a compatible system of representations.
- $\rho_w$  is absolutely irreducible when restricted to any open subgroup of Gal( $\overline{\mathbf{Q}}/F$ ) for all  $w \in L$ .
- Gal( $\mathbf{\bar{Q}}/F$ )/ $\Delta_w$  is cyclic of order prime to l where l denotes the residual characteristic of w, for all  $w \in L$ .
- $[\operatorname{Gal}(\overline{\mathbf{Q}}/F):\Delta_w] \to \infty \text{ as } w \to \infty.$

Then there exists a set of places P of **Q** of Dirichlet density  $1/[E : \mathbf{Q}]$ , all of which split completely in E, such that, for all  $w \in L$  lying above a place  $l \in P$ :

- (i)  $\overline{\rho}_w(\Delta_w)$  is an m-big subgroup of  $GL_n(\mathbf{F}_l)$ .
- (ii) [ker ad  $\overline{\rho}_w : \Delta_w \cap \ker \operatorname{ad} \overline{\rho}_w] > m$ .

Here, as usual,  $\overline{\rho}_w$  denotes the semi-simplified reduction modulo l of  $\rho_w$ . For the definition of a compatible system and a full set of places, see Section 6.

**Remark 1.4.** The first part of the theorem is a mild generalization, from the setting of bigness to *m*-bigness, of the main result of Snowden-Wiles [3]. The result shall be proved by considering their arguments in the m-bigness setting combined with a slight strengthening of [3, Proposition 4.1] by Proposition 3.1.

The second part of the theorem proves another technical result required for the application of automorphy lifting theorems (see [1]). The proof of this result uses an argument of Barnet-Lamb–Gee–Geraghty–Taylor that originally appeared in [1].

The format of this manuscript mirrors that of Snowden–Wiles [3]. We content ourselves here to remark upon the minor changes to [3] that are needed to obtain the above result.

#### 1.1. Notation

Our notation is as in Snowden–Wiles [3]. More specifically, unless explicitly mentioned otherwise, we adhere to the following conventions. Reductive algebraic groups are assumed connected. A semi-simple algebraic group *G* defined over a field *k* is *simply connected* if the root datum of  $G_{\bar{k}}$  is simply connected. If *S* is a scheme, then a group scheme *G*/*S* is *semi-simple* if it is smooth, affine and its geometric fibers are semi-simple connected algebraic groups.

#### 2. Elementary properties of *m*-bigness

In [3, Section 2], a series of results concerning elementary properties of bigness are demonstrated. We remark that the arguments appearing there trivially generalize to give the following results on *m*-bigness:

**Proposition 2.1.** Let *H* be a normal subgroup of *G*. If *H* satisfies the properties (B2), (B3) and (B4) then *G* does as well. In particular, if *H* is *m*-big and the index [G : H] is prime to *l* then *G* is *m*-big.

**Proposition 2.2.** The group G is m-big if and only if  $k^{\times}G$  is m-big where  $k^{\times}$  denotes the group of scalar matrices in GL(V).

**Proposition 2.3.** Let k'/k be a finite extension, let  $V' = V \otimes_k k'$  and let G be an m-big subgroup of GL(V). Then G is also an m-big subgroup of GL(V').

#### 3. Highly regular elements of semi-simple groups

We recall the norm utilized by Snowden–Wiles [3, §3.2]. Let k be a field, let G/k be a reductive algebraic group and let  $T_{\bar{k}}$  be a maximal torus of  $G \times_k \bar{k}$ . For  $\lambda \in X^*(T_{\bar{k}})$  a weight, one defines  $\|\lambda\| \in \bar{k}$  to be the maximal value of  $|\langle \lambda, \alpha^{\vee} \rangle|$  as  $\alpha$  runs through the roots of  $G \times_k \bar{k}$  with respect to  $T_{\bar{k}}$ . For V a representation of G, one defines  $\|V\|$  to be the maximal value of  $\|\lambda\|$  where  $\lambda$  runs through the weights  $\lambda$  appearing in  $V \otimes_k \bar{k}$ .

The following result is a slight strengthening of [3, Proposition 4.1]:

**Proposition 3.1.** Let k be a finite field of cardinality q, Let G/k be a semi-simple algebraic group, let T be a maximal torus of G defined over k, let m and n be positive integers and assume that q is large compared to dim G, n and m. Then, there exists an element  $g \in T(k)$  for which the map

$$\left\{\lambda \in X^*(T_{\bar{k}}): \|\lambda\| < n\right\} \to \bar{k}^{\times}, \quad \lambda \mapsto \lambda(g)^m$$

is injective.

**Proof.** The proof shall follow that of [3, Proposition 4.1] with the difference that we are considering here characters of the form  $\lambda^m$  instead of  $\lambda$ .

To begin let  $S := \{\lambda \in X^*(T_{\bar{k}}): \lambda \neq 1, \|\lambda\| < 2n\}$ . We claim that

$$T(k) \not\subset \bigcup_{\lambda \in S} \ker \lambda^m$$

This is equivalent to the statement

$$T(k) \neq \bigcup_{\lambda \in S} \ker \lambda^m \cap T(k)$$

The later statement shall be proved by considering the cardinality of the two sides. Firstly, by [3, Lemma 4.2],  $|T(k)| \ge (q-1)^r$  where *r* denotes the rank of *T*. Consider now the right-hand side. We remark that for  $\lambda \in X^*(T_{\bar{k}})$ ,

$$|\ker \lambda^m \cap T(k)| \leq R_{m,a} |\ker \lambda \cap T(k)|$$

where  $R_{m,q}$  denotes the cardinality of the kernel of the map  $k^{\times} \to k^{\times}$ ,  $k \mapsto k^m$ . Furthermore, we can ensure that  $R_{m,q}/q$  is as small as desired simply by considering q sufficiently large with respect to m. We can now bound the cardinality of the right-hand side by  $NR_{m,q}M$  where the terms are defined as follows.

- N is equal to the cardinality of S, which by [3, Lemma 4.3] is bounded in terms of dim G and n.
- *M* is equal to the maximum cardinality of ker  $\lambda \cap T(k)$  for  $\lambda \in S$ , which by [3, Lemma 4.4] is bounded by  $C(q+1)^{r-1}$  for some constant *C* depending only upon dim *G* and *n*.

Thus for q sufficiently large with respect to dim G, n and m, the cardinality of the right-hand side is strictly less than that of the cardinality of the left-hand side and the claim follows.

As such we can choose a  $g \in T(k)$  such that  $g \notin \ker \lambda^m$  for all  $\lambda \in S$ . Then, for all  $\lambda, \lambda' \in X^*(T_{\bar{k}})$  such that  $\lambda \neq \lambda'$ ,  $\|\lambda\| < n$  and  $\|\lambda'\| < n$ , we have that  $\lambda - \lambda' \in S$  and it follows that  $\lambda(g)^m \neq \lambda'(g)^m$ .  $\Box$ 

#### 4. m-Bigness for algebraic representations

We show here that [3, Proposition 5.1] naturally generalizes to the setting of *m*-bigness.

**Proposition 4.1.** Let *m* be a positive integer, let *k* be a finite field, let *G*/*k* be a reductive algebraic group and let  $\rho$  be an absolutely irreducible representation of *G* on a *k*-vector space *V*. Assume that the characteristic of *k* is large compared to *m*, dim *V* and ||V||. Then  $\rho(G(k))$  is an *m*-big subgroup of *GL*(*V*).

**Proof.** Firstly, we note that by [3, Proposition 5.1] the conditions (B1), (B2) and (B3) are satisfied. Thus, it only remains to check the condition (B4) (in the *m*-bigness setting). The proof is almost identical to the 1-bigness case (cf. [3, Proposition 4.1]); the sole difference comes from appealing to Proposition 3.1 in lieu of [3, Proposition 4.1].

More specifically, as in [3, Proposition 5.1], one begins by reducing to the case where *G* is semi-simple, simply connected and the kernel of  $\rho$  is finite. Choose a Borel *B* of *G* defined over *k*; this is possible as every reductive group scheme defined over a finite field is quasi-split. Let *T* be a maximal torus of *B* and let *U* be the unipotent radical of *B*. The representation  $V_{\bar{k}} = V \otimes_k \bar{k}$  decomposes via its weights:

$$V_{\bar{k}} = \bigoplus_{\lambda \in S} V_{\bar{k},\lambda}$$

where *S* denotes the set of weights of  $(G_{\bar{k}}, T_{\bar{k}})$ . By Proposition 3.1, we can find a  $g \in T(k)$  such that

$$\lambda(g)^m \neq \lambda'(g)^m$$
 for all distinct  $\lambda, \lambda' \in S$ 

We remark that (ignoring multiplicity) the eigenvalues of g in  $V_{\bar{k}}$  are equal to  $\{\lambda(g): \lambda \in S\}$ . It follows that the generalized g-eigenspaces are equal to the weight spaces:

$$V_{\bar{k},g,\lambda(g)} = V_{\bar{k},\lambda}$$
 for all  $\lambda \in S$ 

Let  $\lambda_0$  be the highest weight space (with respect to *B*) and let  $V_{\bar{k},0} := V_{\bar{k},\lambda_0}$  be the corresponding highest weight space. In fact  $V_{\bar{k},0} = V^U \otimes_k \bar{k}$  and as such  $\lambda_0(g) \in k$ . By [3, Proposition 3.7],  $V_{\bar{k},0}$  is 1-dimensional. That is,  $\lambda_0(g)$  is a simple root of  $h_g$ , the characteristic polynomial of *g*. Furthermore, by the properties of *g*, the *m*-th powers of the roots of  $h_g$  are distinct.

Finally it is shown in the proof of [3, Proposition 5.1] that for each irreducible *G*-submodule *W* of ad *V*, there exists a  $f \in W$  such that the composite

$$V_{g,\lambda_0(g)} \hookrightarrow V \xrightarrow{f} V \twoheadrightarrow V_{g,\lambda_0(g)}$$

is non-zero. 🗆

#### 5. *m*-Bigness for nearly hyperspecial groups

Let *l* be a rational prime, let  $K/\mathbf{Q}_l$  be a finite field extension, let  $\mathcal{O}_K$  be the ring of integers and let *k* be the residue field. For *G* an algebraic group over *K*, we define the following *K*-algebraic groups:

- $G^{\circ}$ : the connected identity component.
- $G^{ad}$ : the adjoint algebraic group, which is the quotient of  $G^{\circ}$  by its radical.
- $G^{sc}$ : the simply connected algebraic group cover of  $G^{ad}$ .

We have the natural maps:

 $G \xrightarrow{\sigma} G^{\mathrm{ad}} \xleftarrow{\tau} G^{\mathrm{sc}}$ 

Following Snowden–Wiles [3], we shall call a subgroup  $\Gamma \subset G(K)$  nearly hyperspecial if  $\tau^{-1}(\sigma(\Gamma))$  is a hyperspecial subgroup of  $G^{sc}(K)$ . **Proposition 5.1.** Let *m* be a positive integer, let  $\Gamma$  be a profinite group and let  $\Delta \subset \Gamma$  be an open normal subgroup. Let  $\rho : \Gamma \to GL_n(K)$ be a continuous representation and let G be the Zariski closure of its image. Assume that the following properties are satisfied:

- The characteristic l of k is large compared to n and m.
- The restriction of  $\rho$  to any open subgroup of  $\Gamma$  is absolutely irreducible.
- The index of  $G^{\circ}$  in G is small compared to l.
- The subgroup  $\rho(\Gamma) \cap G^{\circ}(K)$  of  $G^{\circ}$  is nearly hyperspecial.
- $\Gamma/\Delta$  is cyclic of order prime to l.

Then the following holds:

- $-\overline{\rho}(\Delta)$  is an m-big subgroup of  $GL_n(k)$ .
- There exists a constant C depending only upon n such that

[ker ad  $\overline{\rho}$  :  $\Delta \cap$  ker ad  $\overline{\rho}$ ] > [ $\Gamma$  :  $\Delta$ ]/C

**Proof.** Let us remark that the first statement is proved in almost the same way as the proof of [3, Proposition 6.1]. There are two minor differences, firstly we appeal here to Proposition 4.1 instead of [3, Proposition 5.1] and secondly we use an argument of Barnet-Lamb-Gee-Geraghty-Taylor to deduce the *m*-bigness of  $\overline{\rho}(\Delta)$  instead of  $\overline{\rho}(\Gamma)$ . The proof of the second statement is due to Barnet-Lamb-Gee-Geraghty-Taylor and originally appeared in [1].

Let  $\Gamma^{\circ} = \rho^{-1}(G^{\circ})$  and let  $\Delta^{\circ} = \Delta \cap \Gamma^{\circ}$ . Then  $\overline{\rho}(\Delta^{\circ})$  is a normal subgroup of  $\overline{\rho}(\Delta)$  and its index divides  $[G:G^{\circ}][\Gamma:\Delta]$ . which, by assumption, is prime to l (recall l is sufficiently large with respect to  $[G:G^{\circ}]$ ). Thus, by Proposition 2.1, to prove that  $\overline{\rho}(\Delta)$  is *m*-big it suffices to prove that  $\overline{\rho}(\Delta^{\circ})$  is *m*-big. Similarly, to prove the second part of the theorem it clearly suffices to prove the analogous statement for  $\Gamma^{\circ}$  and  $\Delta^{\circ}$ . As such, we can now assume that  $G = G^{\circ}$ .

Let  $V = K^n$  be the representation space of  $\rho$ . By [3, Lemma 6.3], we can find the following:

- A  $\Gamma$ -stable lattice  $\Lambda$  in V.
- A semi-simple group scheme  $\widetilde{G}/\mathcal{O}_K$  whose generic fiber is equal to  $G^{sc}$ .
- A representation  $r: \widetilde{G} \to GL(\Lambda)$  which induces the natural map  $G^{sc} \to G$  on the generic fiber.

These objects can be chosen such that

•  $\mathcal{O}_K^{\times} \cdot r(\widetilde{G}(\mathcal{O}_K))$  is an open normal subgroup of  $\mathcal{O}_K^{\times} \cdot \rho(\Gamma)$ , whose index can be bounded by a constant C defined in terms of *n*.

Furthermore, the generic fiber of r is necessarily an absolutely irreducible representation of  $\widetilde{G}_K$  on V. By [3, Proposition 3.5],  $\Lambda \otimes_{\mathcal{O}_K} k$  is an absolutely irreducible representation of  $\widetilde{G} \times_{\mathcal{O}_K} k$  and its norm is bounded in terms of n. Now  $\widetilde{G} \times_{\mathcal{O}_K} k$  is a semi-simple, simply connected, algebraic group and hence a finite product of simple, simply connected, k-algebraic groups. As l > 4, we have that  $\widetilde{G}(k)$  is perfect (cf. [4]). It follows, as  $\Delta$  is a normal subgroup of  $\Gamma$ whose quotient is abelian, that we have the following chain of normal subgroups:

$$k^{\times}r\big(\widetilde{G}(k)\big) \leqslant k^{\times}\overline{\rho}(\Delta) \leqslant k^{\times}\overline{\rho}(\Gamma)$$

Furthermore,  $[k^{\times}\overline{\rho}(\Gamma):k^{\times}r(\widetilde{G}(k))] < C$ . The second part of the theorem is now immediate.

The first part of the theorem is proved as follows. Proposition 4.1 implies that  $r(\tilde{G}(k))$  is m-big. Then, Propositions 2.2 and 2.1 allow one to deduce that  $\overline{\rho}(\Delta)$  is *m*-big.  $\Box$ 

#### 6. *m*-Bigness for compatible systems

**Definition 6.1.** A group with Frobenii is a triple  $(\Gamma, \mathcal{P}, \{\mathcal{F}_{\alpha}\}_{\alpha \in \mathcal{P}})$  where  $\Gamma$  is a profinite group,  $\mathcal{P}$  is an index set and  $\{\mathcal{F}_{\alpha}\}_{\alpha \in \mathcal{P}}$ is a dense set of elements of  $\Gamma$ . The  $\mathcal{F}_{\alpha}$  are called the *Frobenii* of the group.

**Remark 6.2.** If F is a number field then the corresponding global Galois group  $Gal(\overline{\mathbf{0}}/F)$  is naturally a group with Frobenii.

**Definition 6.3.** A compatible system of representations (with coefficients in a number field *E*) is a triple  $(L, \mathcal{X}, \{\rho_{\lambda}\}_{\lambda \in L})$  where L is a set of places of E,  $\mathcal{X} \subset \mathcal{P} \times L$  is a subset and each  $\rho_{\lambda} : \Gamma \to GL_n(E_{\lambda})$  is a continuous representation, such that the following conditions are satisfied:

- For all  $\alpha \in \mathcal{P}$ , the set  $\{\lambda \in L: (\alpha, \lambda) \notin \mathcal{X}\}$  is finite.
- For all finite sets of places  $\lambda_1, \ldots, \lambda_k \in L$ , the set  $\bigcap_{i=1}^k \{\mathcal{F}_\alpha : (\alpha, \lambda_i) \in \mathcal{X}\}$  is dense in  $\Gamma$ .
- For all  $(\alpha, \lambda) \in \mathcal{X}$ , the characteristic polynomial of  $\rho_{\lambda}(\mathcal{F}_{\alpha})$  has coefficients in *E* and depends only upon  $\alpha$ .

The set of places *L* is said to be *full* if there exists a set *L'* of rational primes of Dirichlet density 1 such that for all places  $\lambda$  of *E* lying above an  $l \in L'$ , we have that  $\lambda \in L$ .

The main theorem can now be stated. It is a mild generalization of [3, Theorem 8.1] and is proved in the same way by simply appealing to Proposition 5.1 instead of [3, Proposition 6.1].

**Theorem 6.4.** Let *m* be a positive integer, let  $(\Gamma, \mathcal{P}, \{\mathcal{F}_{\alpha}\}_{\alpha \in \mathcal{P}})$  be a group with Frobenii, let *E* be a Galois extension of **Q** let *L* be a full set of places of *E* and for each  $w \in L$ , let  $\rho_w : \Gamma \to GL_n(E_w)$  be a continuous representation and let  $\Delta_w \subset \Gamma$  be a normal open subgroup. Assume the following properties are satisfied:

- The  $\rho_w$  form a compatible system of representations.
- $\rho_w$  is absolutely irreducible when restricted to any open subgroup of  $\Gamma$  for all  $w \in L$ .
- $\Gamma/\Delta_w$  is cyclic of order prime to l where l denotes the residual characteristic of w.

-  $[\Gamma : \Delta_w] \to \infty$  as  $w \to \infty$ .

Then there exists a set of places P of **Q** of Dirichlet density  $1/[E : \mathbf{Q}]$ , all of which split completely in E, such that, for all  $w \in L$  lying above a place  $l \in P$ :

- (i)  $\overline{\rho}_w(\Delta_w)$  is an *m*-big subgroup of  $GL_n(\mathbf{F}_l)$ .
- (ii) [ker ad  $\overline{\rho}_w$ :  $\Delta_w \cap \ker \operatorname{ad} \overline{\rho}_w$ ] > *m*.

Theorem 1.3 is then the special case of the above theorem where  $\Gamma$  is the absolute Galois group of a number field.

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