Mathematical Analysis

How likely is Buffon’s ring toss to intersect a planar Cantor set?

Quelles sont les chances pour un cercle de Buffon lancé sur le plan de faire l’intersection avec une voisinage d’un ensemble de Cantor?

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ABSTRACT

In Bateman and Volberg (2008) [1], it was shown that the $n$-th partial $1/4$ Cantor in the plane set decays in Favard length no faster than $C \log n$. In Bond and Volberg (2008) [2], the so-called circular Favard length of the same set is studied, and the same estimate is shown to persist when the circle has radius $r \geq Cn$. By considering characteristic functions, the result of Bond and Volberg (2008) [2] naturally leads to a conjecture which (if true) would imply the sharpness of the $L \log \log L$ boundedness of the circular maximal operator proved by Seeger, Tao and Wright (2005) [3].

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1. Definitions

The four-corner Cantor set $K$ is constructed by replacing the unit square by four sub-squares of side length 1/4 at its corners, and iterating this operation in a self-similar manner in each sub-square. After the $n$th iteration of the similarity maps, let us call the resulting set $K_n$.

The Favard length, or Buffon needle probability, of a planar set $E$ is defined by

$$\text{Fav}(E) = \frac{1}{\pi} \int_0^\pi \left| \text{Proj}_\theta(E) \right| d\theta,$$

(1)

where $\text{Proj}_\theta$ denotes the orthogonal projection from $\mathbb{R}^2$ to direction with angle $\theta$, and $|A|$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

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In [2], a related circular Favard length, or Buffon noodle probability, was studied. To get circular Favard length \( \text{Fav}_r \) instead of usual Favard length \( \text{Fav} \), orthogonal projection along the line is replaced by projection along a circular arc tangent to the line. Specifically, define

\[
F_r(y) := r - \sqrt{r^2 - y^2}.
\]

(2)

Also define \( \sigma_0(x, y) := (x - F_r(y), y) \), and \( \sigma_\theta := R_{-\theta} \circ \sigma_0 \circ R_\theta \), where \( R_\theta \) is clockwise rotation by the angle \( \theta \). Finally, let

\[
\text{Fav}_\sigma(K_n) := \frac{1}{\pi} \int_0^\pi |\text{Proj}_\theta(\sigma_\theta(K_n))| \, d\theta.
\]

For any Cantor square \( Q \subset K_n \), let \( \chi_{Q,\sigma} := \chi_{\text{Proj}_\sigma(\sigma_\theta(Q))} \).

2. The result and the main approach

One way of studying Favard length of structured discrete sets like \( K_n \) is through a certain projection multiplicity function \( f_{n,\theta,\sigma} \) (as used in [1,2], and others):

\[
f_{n,\theta,\sigma} := \sum_{\text{Cantor squares } Q \subset K_n} \chi_{Q,\sigma}.
\]

This is because \( \text{Proj}_\theta(\sigma_\theta(K_n)) = \text{supp}(f_{n,\theta,\sigma}) \), which we will also call \( E_{n,\theta,\sigma} \). The idea is that as the similarities are iterated, the squares stack in a self-similar manner, and the \( L^2 \) norms of \( f_{n,\theta,\sigma} \) should grow, while \( |E_{n,\theta,\sigma}| \) should decrease. However, the Cauchy inequality describes a limitation on this effect: for any fixed interval of angles \( I \),

\[
\int_I |E_{n,\theta,\sigma}| \geq \left( \int_I \int_{\mathbb{R}} f_{n,\theta,\sigma} \, dx \, d\theta \right)^2 \left( \int_I \int_{\mathbb{R}} f_{n,\theta,\sigma} \, dx \, d\theta \right). \tag{3}
\]

The idea is to pick \( \approx \log n \) many disjoint intervals \( I_j \) such that each such estimate gives

\[
\int_{I_j} |E_{n,\theta,\sigma}| \, d\theta \geq \frac{C}{n}. \tag{4}
\]

Summing over \( j \), the result will be:

**Theorem 2.1.** For each \( c > 0 \), there exists \( C > 0 \) such that whenever \( r \geq cn \), \( \text{Fav}_\sigma(K_n) \geq C \frac{\log n}{n} \). Further, we may interpret \( \text{Fav}(K_n) \) to be \( \text{Fav}_\sigma(K_n) \) in the case \( r = \infty \).

Good intervals \( I_j \) can be found near \( \theta = \arctan(1/2) \), because on this direction, \( K_n \) orthogonally projects onto a single connected interval, and the projected squares intersect only on their endpoints. These almost-disjoint projected intervals induce a 4-adic structure on the interval. Let us rotate the axes and redefine the old arctan direction.

We will then let \( I_j := [\arctan(4^{-j-1}), \arctan(4^{-j})], \ 3 < j < \log n \). Then \( I_{\log n} \) will be the closest direction to 0, and it’s reasonable to think that on average, each time \( j \) decreases by 1, \( I_j \) will grow by the factor 4, and \( |E_{n,\theta,\sigma}| \) will decay no more than by a factor of \( 1/4 \), resulting in estimate (4).

Trivially, \( \int_I \int_{\mathbb{R}} f_{n,\theta,\sigma} \, dx \, d\theta \leq C 4^{-2j} \), while

\[
\int_{I_j} \int_{\mathbb{R}} f_{n,\theta,\sigma} \, dx \, d\theta \leq C 4^{-j-1} \leq C n^{-1} \leq C 4^{-2j} \leq C 4^{-j-1} \leq C 4^{-j} \leq C 4^{-j} \leq C 4^{-j}.
\]

(5)

\footnote{Note that if we replace \( \sigma \) with the identity map, we are in the setting of [1]. We will often appeal to the \( \sigma = \text{id} \) case for intuition, while noting that the content of [2] is that the arguments of [1] carry over into [2] when \( cn \leq r < \infty \) with the only difference being a change in the universal constants.}
We can think of $4^{-J}$ as being $\tan(\theta)$ for $\theta$ such that the $\sigma_\theta$ images of the squares $Q, Q'$ have overlap in the projection onto $\theta$. In [1], it was proved that

$$|A_{j,k}| \leq C4^{2n-k-2j},$$

(6)

when $r = \infty$. To get the same estimate for $cn \leq r < \infty$ as shown in [2], it suffices to compare the two cases with an application of the following lemma:

**Lemma 2.2.** Let $\varepsilon > 0$ be small enough. Let $T : \mathbb{C} \to \mathbb{C}$ be such that $\text{Lip}(T - \text{Id}) < \varepsilon$. Then $\forall z,w \in \mathbb{C}$,

$$|\arg(z - w) - \arg(T(z) - T(w))| < 2\varepsilon (\text{mod } 2\pi).$$

This is where the condition $r > cn$ is used: to make the lemma sufficient for the purposes of relation 5. For any $P = (Q, Q') \in A_{j,k}$, it suffices to have the integral $v_P := \int_{Q} X_{\theta} X_{Q',\theta} \, d\theta$ satisfy the estimate

$$v_P \leq C4^{4-2n},$$

since the integrand is supported only for angles belonging to $I_{j-1}, I_j$, and $I_{j+1}$. So we fix $j$ and sum over $k$ to get

$$\int_{l_{j-1}} \sum_{Q \neq Q'} X_{Q,\theta} X_{Q',\theta} \, d\theta \, dx$$

$$\leq \sum_{n-j+1} \max_{k=1} \{v_P : P \in A_{j,k} \text{ for some } j' = j - 1, j, j + 1\} (|A_{j-1,k}| + |A_{j,k}| + |A_{j+1,k}|) \leq Cn4^{-2j}.$$  

Estimate (7) is elementary when $r = \infty$. When $cn \leq r < \infty$, we exploit a relationship between circular Favard length and the area of the set of centers of the intersecting arcs, i.e., $(r+x) \, d\theta \, dx \approx r \, d\theta$ implies that $v_P \approx \frac{1}{2} |A|$, where $A$ is the intersection of two annuli centered at $r$ and $q'$, both having inner radius $r - 4^{-n}$ and outer radius $r + 4^{-n}$. One can bound $A$ by a rectangle and get the desired estimate by the Mean Value Theorem, for example. This concludes the proof of Theorem 2.1.

3. Sharpness of the $L \log L$ bound on the circular maximal operator

Let $c_n(z) := \{z : |z - \xi| = 4^{-m}\}$, and $Mf(z) := \sup_{m \geq 0} 4^m \int_{c_n(z)} |f(\xi)||d\xi|$. In [3], it was proved that $\max |\log L \to L^1,\infty \text{ is bounded, and then suggested that Favard length estimates could prove the sharpness. While this does not seem to be true, it still seems likely that a positive answer may be given by measuring the level sets of $f_{n,\theta,\sigma}$}. Here and in [2], only the set $f_{n,\theta,\sigma} > 1$ was measured.

It is enough to show that for each $\varepsilon > 0$, $M : L(\log L)^{1-\varepsilon} \to L^1,\infty$ is not bounded, which follows if one can construct sets $E_n$ such that $|E_n| \ll 1$, such that $\max_{\theta} t([z : M\chi_{E_n}(z) > t]) \gg |E_n|^{|\log \log \frac{1}{|E_n|}|}{1-\varepsilon}$.

The idea: $m < n$ will vary. Take a contraction $\widetilde{E}_n$ of $E_n$ (by the factor $4^{-n}$), and then take an $\varepsilon \approx 4^{-2n}$ neighborhood of this, called $E_n$. On a certain set of distance about $4^{-m}$ from $E_n$, there is a relatively large set of circles of radius $4^{-m}$ which intersect $E_n$, so that on this set, $M\chi_{E_n}$ is relatively large. Note that $|E_n| \approx 4^n \cdot 4^{-4m} = 4^{-3n}$, so that $\log |E_n| \approx \log n$.

Let $\mu_{n,m} := \{z : M\chi_{E_n} \geq 4^{m-2n}/(2\pi)\}$. Let $H_{n,m} := \{z : c_n(z) \cap \widetilde{E}_n \neq \emptyset\}$. Then $H_{n,m} \subset \mu_{n,m}$. $H_{n,m} \cap H_{n,m'} = \emptyset$ for $m \neq m'$, and $|H_{n,m}| \geq C4^{-m} \text{Fav}_{\theta_n}(\widetilde{E}_n) \geq C \log n 4^{-n-m}$.

Thus $|\bigcup_{m=0}^n H_{n,m}| \geq \sum_{m=0}^n |H_{n,m}| \geq \sum_{m=0}^n C \log n 4^{-n-m}$. It would be nice if we could instead write the following for, say, $M = an$, for some constant $\alpha > 0$:

$$\{z : M\chi_{E_n} \geq 4^{m-2n}/(2\pi)\} \geq C \alpha \log n 4^{-n} 4^{-M},$$

(8)

then because

$$\frac{4^{M-2n}}{2\pi} \left| \{z : M\chi_{E_n} \geq 4^{M-2n}/(2\pi)\} \right| \geq C4^{-3n} \log n \geq C|E_n| \log \log \frac{1}{|E_n|} \gg |E_n| \left( \frac{1}{|E_n|} \right)^{1-\varepsilon}.$$
Let us state how one might get this. We can call by $Q_j$ ($j = 1, \ldots, 4^n$), the squares composing $\tilde{E}_n$, and let $H_{n,m,M} := \{z: \#j: c_m(z) \cap Q_j \neq \emptyset \geq 4^{M-m}\}$. Then $H_{n,m,M} \subset \mu_{n,M}$.

Relation (8) would then follow if we had $|H_{n,m,M}| \geq C \frac{\log n}{n} 4^{-n-M}$. So we have this strong conjecture:

There exist $\alpha, C > 0$ such that for infinitely many $n$, $|\{(x, \theta) \in \mathbb{R} \times [0, 2\pi]: f_{n,\theta,\sigma} (x) \geq 4^m\}| \geq C \frac{\log n}{n} 4^{-m}$ for all $m \leq \alpha n$.

Alternately, a weak conjecture:

For all $\varepsilon > 0$, there exist $C > 0$ so that if

$\nu(n) := \# \{m \leq n: \#(x, \theta) \in \mathbb{R} \times [0, 2\pi]: f_{n,\theta,\sigma} (x) \geq 4^m\} \geq C \frac{(\log n)^{1-\varepsilon}}{n} 4^{-m}\}$,

then $\limsup \frac{\nu(n)}{n} (\log n)^\varepsilon > 0$.

References