Dynamical Systems

Geometry of the common dynamics of Pisot substitutions with the same incidence matrix

Géométrie commune des substitutions Pisot de même matrice d’incidence

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1. Introduction

Let \( A \) be a finite set of cardinality \( d \). A substitution \( \sigma \) on \( A \) is a morphism of the free monoid \( A^* \) such that the image of each letter of \( A \) is a nonempty word. It naturally extends to the set of one-sided sequences, denoted by \( A^N \).

Let \( f : A^* \mapsto \mathbb{Z}^d : w \mapsto (|w|_i)_{i \in A} \) be the natural homomorphism obtained by abelianization of the free monoid, called the abelianization map. We associate to every substitution \( \sigma \) its incidence matrix \( M \) obtained by abelianization, \( M_{i,j} = |\sigma(j)| \). A substitution \( \sigma \) is primitive if there exists an integer \( k \) such that \( M^k > 0 \). We say that \( \sigma \) is an irreducible Pisot substitution if there exists one eigenvalue of \( M \) which is strictly greater than 1 and all other eigenvalues are strictly less than 1 in modulus. An equivalent definition is that the largest eigenvalue is a Pisot number, and the characteristic polynomial is irreducible. A substitution is unimodular if the determinant of its incidence matrix is equal \( \pm 1 \).

Let \( \sigma \) be a primitive substitution, then there exists a finite number of periodic points \( u \). We associate to \( u \) a symbolic dynamical system \((\Omega_u, S)\) where \( S \) is the shift map on \( A^N \) defined by \( S(a_0 a_1 \cdots) = a_1 a_2 \cdots \) and \( \Omega_u \) is the closure of \( \{S^m(u) : m \geq 0\} \) in \( A^N \).
In the case of unimodular Pisot substitutions, the symbolic dynamical system can be understood in a geometrical way. In [5], G. Rauzy proved that the dynamical system generated by the substitution \( \sigma(1) = 12, \sigma(2) = 13, \sigma(3) = 1 \), is measure-theoretically conjugate to an exchange of domains in a compact set \( \mathcal{R} \) of the complex plane with a self-similar structure. The construction goes as follows:

Using the abelianization map, to any finite or infinite word \( u \), we can associate a canonical stepped line in \( \mathbb{R}^d \) as a sequence \( \langle f(P_k) \rangle \), where \( P_k \) is the prefix of length \( k \) of \( u \). The canonical stepped line associated with a fixed point of a Pisot substitution remains within a bounded distance to the expanding direction (given by the right Perron–Frobenius eigenvector of \( M \)). Projecting the vertices of this canonical stepped line by the projection \( \pi \) on the contracting subspace \( E_i \) of the incidence matrix of \( \sigma \) along its expanding direction, and taking the closure of this set, yields the Rauzy fractal [2]. The projection of the stepped line can be seen as a map \( \pi \) from the orbit \( S^0(\{u\}) \) of the fixed point to the Rauzy fractal. This map can be extended by continuity to a map \( \pi : X_\sigma \rightarrow \mathcal{R} \), see [3]. In many cases, this map is known to be a measurable isomorphism.

The Rauzy fractal is a nonempty compact set which is the closure of its interior and decomposes in a natural way in \( d \) subtils \( \mathcal{R}(1), \ldots, \mathcal{R}(d) \). It is also known that these tiles induce a multiple tiling of the contracting plane. For more detail see [6, Chap. 3].

**Definition 1.1.** An exclusive inner point of the Rauzy fractal is a point from one subtile \( \mathcal{R}_i \), \( i \in \{1, \ldots, d\} \), which does not belong to any other tile from the multiple tiling. See [6, Chap. 4].

We can generalize the definition of the Rauzy fractal with the projection method:

**Definition 1.2.** A substitutive set is the closure of the projection of a canonical stepped line associated with a primitive substitution \( \sigma \) on a contracting space of the incidence matrix of \( \sigma \). See [1].

To a Pisot matrix, there correspond many substitutions since there are many words with the same abelianization, see [7]. A classic example is given by the Tribonacci substitution and the flipped Tribonacci substitution, i.e.,

\[
\sigma_1 : \begin{cases} 
    a \rightarrow ab \\
    b \rightarrow ac \\
    c \rightarrow a
\end{cases} \quad \text{and} \quad \sigma_2 : \begin{cases} 
    a \rightarrow ab \\
    b \rightarrow ca \\
    c \rightarrow a
\end{cases}
\]

The Rauzy fractal of the first substitution is simply connected [5], while it is a well known fact that the second fractal is not simply connected [4].

In this Note we study the common dynamics of two irreducible and unimodular Pisot substitutions \( \sigma_1 \) and \( \sigma_2 \) having the same incidence matrix (see Fig. 1). We show a geometric realization of the intersection of the interior of the two Rauzy fractals associated with \( \sigma_1 \) and \( \sigma_2 \). The main result of this Note is the following:

**Theorem 1.3.** Let \( \sigma_1 \) and \( \sigma_2 \) be two irreducible and unimodular Pisot substitutions with the same incidence matrix. We denote \( \mathcal{R}_1, \mathcal{R}_2 \) their two associated Rauzy fractals. We suppose that 0 is an exclusive inner point of \( \mathcal{R}_1 \). Then the closure of the intersection of the interiors of \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) has strictly positive measure and is a substitutive set.

2. Morphism generating the common points of two Pisot substitutions

Let \( \sigma_1 \) and \( \sigma_2 \) be two unimodular and irreducible Pisot substitutions with the same incidence matrix and \( \mathcal{R}_1, \mathcal{R}_2 \) their two associated Rauzy fractals. We denote by \( \mathcal{R} \) the closure of the intersection of the interiors of \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \). Let \( (\Omega_{\sigma_1}, S) \) and \( (\Omega_{\sigma_2}, S) \) be the symbolic dynamical systems associated with \( \sigma_1 \) and \( \sigma_2 \). We consider \( \pi_1 \) (resp. \( \pi_2 \)) the projection map from the symbolic dynamical system \( (X_{\sigma_1}, S) \) (resp. \( (X_{\sigma_2}, S) \)) on the corresponding Rauzy fractal.

**Lemma 2.1.** Suppose that 0 is an inner point of \( \mathcal{R}_1 \). Then \( \mathcal{R} \) has non-empty interior and strictly positive Lebesgue measure.

**Proof.** Because 0 is an inner point of \( \mathcal{R}_1 \), there exists an open set \( U \) such that \( 0 \in U \subset \mathcal{R}_1 \). The Rauzy fractal is the closure of its interior [6] and 0 is a point of \( \mathcal{R}_2 \), hence there exists a sequence of points \( (x_n)_{n \in \mathbb{N}} \) from the interior of \( \mathcal{R}_2 \) which converges to 0. Then there exist open sets \( V_n \) such that \( x_n \in V_n \subset \mathcal{R}_2 \). Since \( (x_n) \) converges to 0, there exists \( N \in \mathbb{N} \) such that \( x_N \in U \).

The open set \( U \cap V_N \) is non-empty and \( U \cap V_N \subset \mathcal{R}_1 \cap \mathcal{R}_2 \). This implies that \( \mathcal{R} \) contains a non-empty open set, hence it has strictly positive Lebesgue measure. \( \square \)

We denote by \( \Gamma \) the group \( \{ \sum_{i=1}^d n_i e_i : \sum_{i=1}^d n_i = 0, n_i \in \mathbb{Z} \} \subset \mathbb{Z}^d \) where \( (e_1, \ldots, e_d) \) denotes the canonical base of \( \mathbb{R}^d \). We introduce the following lemma from [6, Chap. 4].
Fig. 1. Rauzy fractals of $\sigma_1$, $\sigma_2$ and their intersection.
Lemma 2.2. Let \( 0 \) be an exclusive inner point of \( \mathcal{R}_1 \) the Rauzy fractal associated to an irreducible Pisot substitution. Then \( \mathcal{R}_1 \) is a fundamental domain for the projection of \( \Gamma \) on the contracting plane along the expanding line.

Lemma 2.3. Suppose that \( 0 \) is an exclusive inner point of \( \mathcal{R}_1 \). Let \( W \) be a non-empty open set in \( \mathcal{R} \), define \( V_1 := \pi_1^{-1}(W) \subset \mathcal{O}_{\sigma_1} \) and \( V_2 := \pi_2^{-1}(W) \subset \mathcal{O}_{\sigma_2} \). For any \( y \in W \), such that \( y = \pi_1(v_1) = \pi_2(v_2) \), any return time of \( v_2 \) to \( V_2 \) is a return time of \( v_1 \) to \( V_1 \).

Proof. We consider \( v_1 \in V_1 \) and \( v_2 \in V_2 \) such that \( \pi_1(v_1) = \pi_2(v_2) \). Let \( n \) be a return time of \( v_2 \). By definition, if \( v \) is a fixed point of \( \sigma_1 \), \( \pi_1(S^n v) = \pi_1(f(P_n)) \), where \( P_n \) is a prefix of length \( n \) of \( v \). We can extend this definition by continuity, we obtain \( \pi_1(S^n v_1) = \pi_1(v_1) + \pi_1(f(P_n)) \). Similar \( \pi_2(S^n v_2) = \pi_2(v_2) + \pi_1(f(P_n)) \), where \( P_n \) is a prefix of length \( n \) of \( v_2 \). Since \( \pi_1(S^n v_1) = \pi_2(S^n v_2) = \pi_1(f(P_n)) \), there exists \( w \in \Gamma \) such that \( \pi_1(S^n v_1) = \pi_2(S^n v_2) = \pi_1(w) \).

By hypothesis \( S^n v_2 \in V_2 \) and \( \pi_2(S^n v_2) \) is an inner point of \( \mathcal{R}_1 \). This implies that \( \pi_2(S^n v_2) \) is an inner point of \( \mathcal{R}_1 \). Since \( \pi_1(S^n v_1) \in \mathcal{R}_1 \) and by hypothesis \( \mathcal{R}_1 \) is a fundamental domain, this means that the interior of \( \mathcal{R}_1 \) cannot meet \( \mathcal{R}_1 + \pi_1(w) \), unless \( \pi_1(w) = 0 \). So we have \( \pi_1(S^n v_1) = \pi_2(S^n v_2) \). Hence if \( n \) is a first return time to \( V_2 \), \( n \) is a return time to \( V_1 \).

Definition 2.4. Let \( U \) and \( V \) be two finite words. We say that \( (U, V) \) is a balanced pair if \( f(U) = f(V) \), where \( f : A^* \to \mathbb{Z}^d \) is the abelianization map.

Definition 2.5. A minimal balanced pair is a balanced pair \( (U, V) \) such that for every strict prefixes \( U_k \) of \( U \) and \( V_k \) of \( V \) of the same length \( k \), \( f(U_k) \neq f(V_k) \).

Lemma 2.6. Let \( u \) and \( v \) be two fixed points of \( \sigma_1 \) and \( \sigma_2 \) respectively. We can decompose the double sequence \( (u, v) \) into a finite set of minimal balanced pairs.

Proof. By definition, \( \pi_1(u) = \pi_2(v) = 0 \). Since \( \mathcal{O}_{\sigma_2} \) is a minimal system and \( \pi_2^{-1}(\mathcal{R}) \in \mathcal{O}_{\sigma_2} \) then the return time \( n_k \) to \( \pi_2^{-1}(\mathcal{R}) \) exists. From Lemma 2.3, \( n_k \) is a return time to \( \pi_1^{-1}(\mathcal{R}) \), which implies that \( S^n(U) \in \pi_1^{-1}(\mathcal{R}) \). This means that there exist two prefixes \( U \) and \( V \) of \( u \) and \( v \) respectively, such that \( f(U) = f(V) \). The balanced pair \( (U, V) \) can be decomposed into minimal balanced pairs. We consider the image of each of these minimal pairs by \( \sigma_1 \) and \( \sigma_2 \). Each minimal balanced pair will appear, we consider the image of each new pair by \( \sigma_1 \) and \( \sigma_2 \) and iterate. Since \( (\Omega_{\sigma_1}, S) \) is a minimal system, the first return time to \( \pi_1^{-1}(\mathcal{R}) \) is bounded, so the length of the minimal balanced pairs is bounded and all the minimal balanced pairs will appear after finite time.

Proof of Theorem 1.3. Let us prove now that these common points can be obtained as the projection of a fixed point of a new substitution defined on the set of the minimal balanced pairs. We give an algorithm to obtain this morphism. From Lemma 2.1, there exist two finite words \( W_1 \) and \( W_2 \) prefixes of \( u \) and \( v \) respectively such that \( f(W_1) = f(W_2) \). We can decompose the balanced pair \( (W_1, W_2) \) into minimal balanced pairs. Let \( (v_1, v_2) \) be the first minimal balanced pair. Then, \( (\sigma_1(v_1), \sigma_2(v_2)) \) is a balanced pair because \( \sigma_1 \) and \( \sigma_2 \) have the same matrix. We consider the decomposition of this balanced pair into minimal balanced pairs. This means that we can write \( (\sigma_1(v_1), \sigma_2(v_2)) = (u_1 \cdots u_k, w_1 \cdots w_k) \) where \( f(u_1) = f(w_1), \ldots, f(u_k) = f(w_k) \).

Since the set of common return times is bounded, by iteration with \( \sigma_1 \) and \( \sigma_2 \) we obtain in bounded time the set of all minimal balanced pairs. We can define the substitution \( \Sigma \) over the finite set of minimal balanced pairs: \( \Sigma : (U, V) \mapsto (\sigma_1(U), \sigma_2(V)) \). The set \( \mathcal{R} \) is obtained as the closure of the projection of the stepped line associated to the fixed point of \( \Sigma \). The interior is clearly substitutive with respect to the substitution \( \Sigma \).

Remark 1. We do not know if, under the hypotheses of the theorem, the set \( \mathcal{R}_1 \cap \mathcal{R}_2 \) is the closure of intersection of the interiors of \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \).

Remark 2. We can give examples where the intersection is reduced to the origin, which is in this case an extremal point of the Rauzy fractal.

References