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# Harmonic Analysis

# The Kahane theorem for nonspherical partial sums of Fourier integrals

# *Le théorème de Kahane pour les sommes partielles non sphériques des intégrales de Fourier*

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#### ABSTRACT

When n > 2 it is well known that the spherical partial sums of *n*-fold Fourier integrals of the characteristic function of a ball diverge at the origin, because of the jump at the boundary of the ball. The relation between convergence properties of spherical partial sums and geometry of discontinuities of the function being expanded was investigated in the well-known paper of Kahane. The most remarkable result, proved by Kahane in this paper, asserts that for the characteristic function of a bounded domain in  $R^3$  the inverse statement is also true: if the surface is analytic and if the spherical Fourier inversion fails at a single point, then the surface must be a sphere and the point must be the center. In this Note we consider nonspherical partial sums, i.e. Fourier integrals under summation over smoothly bounded strongly convex symmetric sets and prove the natural generalization of the Kahane theorem.

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#### RÉSUMÉ

Pour n > 2 on sait que les sommes partielles sphériques des intégrales n-ièmes de la fonction caractéristique d'une boule divergent à l'origine ; cela résulte du saut de cette fonction à la frontière de la boule. La relation entre les propriétés de convergence des sommes partielles sphériques et la géométrie des discontinuités de la fonction considérée a été étudiée en détail dans un article bien connu de Kahane : le résultat le plus intéressant démontré par Kahane est que pour la fonction caractéristique d'un domaine borné de  $R^3$  la proposition réciproque est également vraie, à savoir que si la surface est analytique et si l'inverse de Fourier est réduite à un point, alors la surface doit être une sphère et le point est le centre de cette sphère. Dans cette Note on considère des sommes partielles non sphériques, c'est-à-dire des intégrales de Fourier sur des ensembles symétriques, fortement convexes bornés à frontières régulières ; on démontre ainsi une généralisation naturelle du théorème de Kahane.

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#### 1. Introduction

The main object of this Note is a class of piecewise smooth functions which we define as follows:

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**Definition 1.1.** Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with smooth boundary  $\Gamma$ . A function f(x) is said to be piecewise smooth if  $f(x) = \chi_{\Omega}(x)g(x)$ , where  $\chi_{\Omega}(x)$  is a characteristic function of  $\Omega$ , and  $g(x) \in C^{\infty}(\mathbb{R}^n)$ .

Let C be an open set in  $\mathbb{R}^n$  symmetrical with respect to the origin and with a smooth strongly convex boundary. Recall, a surface is called strongly convex if the Gaussian curvature is positive at every point of the surface.

We consider nonspherical partial sums of *n*-fold Fourier integrals associated with *C*,

$$S_{\lambda C} f(x) = \int_{\lambda^{-1} \xi \in C} \hat{f}(\xi) e^{ix \cdot \xi} d\xi, \qquad (1)$$

where  $\hat{f}(\xi)$  is the Fourier transform of a piecewise smooth function  $f: \hat{f}(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx$ . If C is a ball,  $C = \{\xi: |\xi| < 1\}$ , then we have the spherical partial sums.

If we introduce a function  $A(\xi) \in C^{\infty}(\mathbb{R}^n \setminus 0)$  with properties:

$$A(t\xi) = tA(\xi), \quad \forall t > 0, \quad \forall \xi \in \mathbb{R}^n, \tag{2}$$

then we may define symmetric, positive elliptic pseudodifferential operator A(D) with domain of definition  $C_0^{\infty}(\mathbb{R}^n)$  and order 1 as follows:  $A(D)u(x) = \int_{\mathbb{R}^n} \widehat{u}(\xi) A(\xi) e^{ix\cdot\xi} d\xi$ , where  $u \in C_0^{\infty}(\mathbb{R}^n)$ . The function  $A(\xi)$  is called a symbol of the pseudodifferential operator A(D). The closure  $\overline{A}$  of A(D) in  $L_2(\mathbb{R}^N)$  is a selfadjoint operator and it can be represented by (see, for example, [15])  $\overline{A} = \int_0^{\infty} \lambda dE_{\lambda}$ , where  $\{E_{\lambda}\}$  is decomposition of unity. It is not hard to show, that  $E_{\lambda}f(x)$  coincides with (1). A very simple example for A(D) is  $\sqrt[m]{P_m(D)}$ , where  $P_m(D)$  is a positive elliptic differential operator of order m.

If n = 2 then the partial sums  $S_{\lambda C} f(x)$  of a piecewise smooth function f as  $\lambda \to \infty$ , converge uniformly on any compact  $K \subset \mathbb{R}^n \setminus \Gamma$  (see, for example, [13] and [1]), no matter how the set C and the set of discontinuity  $\Gamma$  of f are related. But when  $n \ge 3$ , however, this relation is a key factor. For example, simple calculations show, that the spherical partial sums of n-fold Fourier integrals of the characteristic function of a ball diverge at the origin, although the function in question is smooth at this point. In 1993 M. Pinsky [12] studied in detail the convergence at the origin of spherical partial sums for functions supported in the ball  $\{x: |x| \le R\}$  and discovered the criterion for the convergence at the origin. In the mathematical literature this result is called "the Pinsky phenomenon". The relation between convergence properties of spherical partial sums and geometry of discontinuities of the function being expanded was also investigated in the well-known paper of Kahane [8]. The most remarkable result, proved by Kahane in this paper asserts, that for the characteristic function of a bounded domain in  $\mathbb{R}^3$  the inverse of the statement on the characteristic function of the ball is also true: if the surface is analytic and if the spherical Fourier inversion fails at a single point, then the surface must be a sphere and the point must be its center.

If we consider the absolute convergence, then even in case n = 2 the convergence of  $S_{\lambda C} f(x)$  depends on geometrical properties of the relationship between  $\Gamma$  and C. This was for the first time noted and studied by Maslov in [10] and [9]. Following Maslov's idea, Sh.A. Alimov in [2] and [3] classified points in  $\mathbb{R}^n$ ,  $n \ge 3$ , depending on their position to the surface of discontinuity  $\Gamma$  of the piecewise smooth function f and obtained sufficient conditions for the uniform convergence of  $S_{\lambda C} f(x)$ . In particular, the author proved [3], that the spherical partial sums converge at the points that satisfy the following condition: for each sphere centered at any of these points the order of the contact with  $\Gamma$  do not exceed 1. Considering the characteristic function of a unit ball in  $\mathbb{R}^3$ , we can see that the mentioned sufficient condition is quite precise.

It was first noted by Pinsky and Taylor [13] (see also [6]), that in order to obtain the same effect as the Pinsky phenomena for nonspherical partial sums  $S_{\lambda C} f(x)$ , one should consider functions supported in the dual to *C* set.

**Definition 1.2.** Let  $A(\xi)$  be defined by (2) and (3). Then an open set *B* with a smooth boundary  $\partial B$  is said to be dual to *C* if  $\partial B = \{\nabla A(\xi), \xi \in \partial C\}$ .

In [13] a piecewise smooth function  $f_0(x)$  was defined as follows: let  $g(x) \in C^{\infty}(\mathbb{R}^N)$ , then  $f_0(x) = g(x)$ ,  $x \in B$  and  $f_0(x) = \frac{1}{2}g(x)$ ,  $x \in \partial B$  and  $f_0(x) = 0$  otherwise. The authors proved in particular, the following:

**Theorem** (Pinsky and Taylor). Let  $f_0$  be a piecewise smooth function, defined as above, and  $n \ge 3$ . Then  $\lim_{\lambda \to \infty} S_{\lambda C} f_0(x) = f_0(x)$ ,  $\forall x \neq 0$ .

Convergence of the partial sums  $S_{\lambda C} f_0(0)$  recently was investigated in [5]. The authors established necessary and sufficient conditions for  $f_0$ , which guarantee the convergence of this partial sums. In other words, authors proved the Pinsky phenomenon for  $S_{\lambda C} f(x)$ , when a piecewise smooth function f is supported on the dual set B and  $n \ge 3$ . From this result, in particular, one has, that the partial sums  $S_{\lambda C} f(x)$  of the characteristic function f of the dual set B diverge at the origin. After having this result proved, one may ask, whether or not the inverse of it is true as well? The aim of this paper is to give a positive answer to this question, proving the result of Kahane with appropriate formulation for nonspherical partial sums  $S_{\lambda C} f(x)$  of 3-fold Fourier integrals.

#### 2. On dual sets

In this section we follow Alimov's idea in [2] and introduce the p(x) function and use it to give an equivalent definition for dual sets.

Let  $\partial C$  be a closed connected analytic surface in  $\mathbb{R}^n$  with strongly positive Gaussian curvature at each its point. (Throughout the paper we use the term "analytic" just for "R-analytic" objects and do not consider complex functions and surfaces.) It follows from the convexity of  $\partial C$  that for any  $x \in \mathbb{R}^n$ ,  $x \neq 0$  there exists a pair of points  $\theta_+(x)$  and  $\theta_-(x) \in \partial C$  at which the exterior normal to  $\partial C$  has the same direction with x and -x respectively. Note that the mapping  $x \to \theta_+(x)$  is inverse to the Gauss mapping. Since the surface  $\partial C$  is analytic, then using the property of the Gauss mapping, we obtain that functions  $\theta_{\pm}(x)$  are analytic (see, for example, [11, p. 161]). Obviously  $\theta_{\pm}(x) = \theta_{\mp}(-x)$  and it is clear that  $\theta_{\pm}(tx) = \theta_{\pm}(x)$ , t > 0,  $x \in \mathbb{R}^n$ . Now we can define an analytic function on  $\mathbb{R}^n \setminus \{0\}$  as  $p(x) = x \cdot \theta_+(x)$ ,  $x \in \mathbb{R}^n$ . We have  $p(\lambda x) = |\lambda| p(x) \ge 0$ . Note if

 $A(\xi) = |\xi|$ , then p(x) = |x|.

One can prove (see, e.g., [5]), that p(x) satisfies the following properties:

**Proposition 2.1.** There exist positive constants  $c_0$  and  $c_1$  such that (1)  $p(x) \ge c_0|x|$ ; (2)  $|\nabla p(x)| \ge c_1|x|$ .

The following proposition gives us the way to define the notion of dual set by using the introduced function p(x).

**Proposition 2.2.** Let B be the dual to C set and p(x) be defined as above. Then  $B = \{x; p(x) < 1\}$ .

**Proof.** Let us introduce the notation  $\partial G = \{x: p(x) = 1\}$  and  $v(\xi)$  be a unit normal vector to  $\partial C$  at the point  $\xi$ . As it follows directly from the definition  $\partial G = \{\alpha(\xi)\nu(\xi): \xi\nu(\xi) = \frac{1}{\alpha(\xi)}, \xi \in \partial C\}$ , where  $\alpha(\xi) > 0$  is a scalar function. Let  $A(\xi)$  be defined by (2) and (3). Then  $\nu(\xi) = |\nabla A(\xi)|^{-1} \nabla A(\xi)$  and the homogeneity of  $A(\xi)$  implies that  $\xi \nabla A(\xi) = A(\xi)$ , for  $\xi \neq 0$ . Particularly for  $\xi \in \partial C$ , we have  $\xi \nabla A(\xi) = 1$  and hence  $\alpha(\xi) = |\nabla A(\xi)|$ . This fact leads us to  $\partial G = \{\nabla A(\xi) : \xi \in \partial C\} = \partial B$ .  $\Box$ 

It is convenient to introduce the notion of quasi-ball (see [2]):

$$B(x_0; r) = \{x: p(x - x_0) < r\}.$$

Note, for the dual set we have B = B(0; 1).

#### 3. Main result

From here on we consider only three-dimensional case, i.e. n = 3. First, we prove the following technical lemma.

**Lemma 3.1.** Let  $f(x) : \mathbb{R}^3 \to \mathbb{R}$  be a piecewise smooth function as in Definition 1.1. Then for the oscillatory integral  $J(\lambda) = \int_{\Omega} f(x)e^{i\lambda S(x)} dx$ , with a phase function  $S : \nabla S(x) \neq 0$ ,  $\forall x \in \Omega$ , one has:  $J(\lambda) = \frac{1}{i\lambda} \int_{\Gamma} \frac{\partial S(x)}{\partial n(x)} |\nabla S(x)|^{-2} f(x)e^{i\lambda S(x)} d\sigma(x) + O(\lambda^{-2})$ , as  $\lambda \to \infty$ , from here on we use the notation  $d\sigma(x)$  for the surface Lebesgue measure.

**Proof.** We introduce a differential operator,  $M(D) = \frac{1}{|\nabla S(x)|^2} \sum_{j=1}^3 \frac{\partial S(x)}{\partial x_j} \frac{\partial}{\partial x_j}$ . Then  $M(D)(e^{i\lambda S(x)}) = i\lambda e^{i\lambda S(x)}$ , and integrating by parts, one has:  $J(\lambda) = \frac{1}{i\lambda} \int_{\Omega} f(x)M(D)(e^{i\lambda S(x)}) dx = \frac{1}{i\lambda} \int_{\Gamma} \frac{\partial S(x)}{\partial n(x)} |\nabla S(x)|^{-2} f(x)e^{i\lambda S(x)} d\sigma(x) - \frac{1}{i\lambda} \int_{\Omega} L(D)(f(x))e^{i\lambda S(x)} dx$ , where L(D) is a formally conjugate operator to M(D). Integrating by parts once more we obtain that the second integral can be estimated as  $O(\lambda^{-2})$ .

We recall that  $\partial C$  is a closed connected analytic surface in  $R^3$  with strongly positive Gaussian curvature at each point. The Fourier integral partial sums have the form:  $S_{\lambda C} f(x) = (2\pi)^{-3} \int_{\lambda^{-1}\xi \in C} \int_{R^3} f(y) e^{i(x-y)\cdot\xi} dy d\xi$ .

Let f(x) be a piecewise smooth function as in Definition 1.1, and the surface of discontinuity  $\Gamma$  of f be analytic and connected. Then as it follows from the Fubini theorem,  $S_{\lambda C} f(x) = \int_{\Omega} f(y) D_{\lambda}(x-y) dy$ , where the function  $D_{\lambda}(z) = (2\pi)^{-3} \int_{\lambda^{-1} \xi \in C} e^{iz \cdot \xi} d\xi$  is known as the Dirichlet kernel. The Dirichlet kernel for strongly convex  $\partial C$  is well-studied and in particular one has the asymptotic expansion (see [7, p. 117]):

$$D_{\lambda}(z) = C\lambda \frac{|K(\theta_{+}(z))|^{-1/2}}{|z|^{2}} \Big[ e^{i(\lambda p(z) - \frac{\pi}{2})} \big( 1 + \Phi_{+}(z;\lambda) \big) - e^{-i(\lambda p(z) - \frac{\pi}{2})} \big( 1 + \Phi_{-}(z;\lambda) \big) \Big],$$
(5)

where  $C = -i/(2\pi)^{(n+1)/2}$  and  $\Phi_{\pm}(z;\lambda) = \sum_{j=1}^{\infty} c_j^{\pm}(z)\lambda^{-j}$ , with  $c_j^{\pm} \in C^{\infty}$ . Let  $x_0$  be a point in  $\Omega$ . We fix  $r_0$  such that  $B(x_0;r_0) \subset \Omega$  and introduce the function  $\psi(x) \in C_0^{\infty}(\mathbb{R}^3)$ , such that  $\psi(x) = 1$ if  $|x - x_0| \leq r_0/2$  and  $\operatorname{supp} \psi \subset B(x_0; r_0)$ .

We have:

$$S_{\lambda C} f(x_0) = \int_{\Omega} \psi(y) f(y) D_{\lambda}(x_0 - y) \, \mathrm{d}y + \int_{\Omega} \left( 1 - \psi(y) \right) f(y) D_{\lambda}(x_0 - y) \, \mathrm{d}y.$$
(6)

We note that  $\psi(x) f(x)$  is a smooth function in  $\mathbb{R}^3$  and therefore the first integral here converges to  $f(x_0)$  as  $\lambda \to \infty$  (see, e.g., [4]).

Now we introduce the following notation:  $I(\lambda) = \int_{\Omega} (1 - \psi(y)) f(y) D_{\lambda}(x_0 - y) dy$ . Using the translation  $z = y - x_0$  one has:  $I(\lambda) = \int_{\Omega} (1 - \psi(y)) f(y) D_{\lambda}(x_0 - y) dy = \int_{\Omega'} (1 - \psi(z + x_0)) f(z + x_0) D_{\lambda}(z) dz$ , where  $\Omega' = \{z : z = y - x_0, y \in \Omega\}$ .

Employing the expansion (5) we have:  $I(\lambda) = \lambda \int_{\Omega'} G_1(z)(e^{i\lambda p(z)} + e^{-i\lambda p(z)}) dz + \int_{\Omega'} G_2(z)(e^{i\lambda p(z)} + e^{-i\lambda p(z)}) dz$ . It is necessary to note that both functions  $G_1, G_2$  vanish near the origin. Therefore we can apply Lemma 3.1 to all integrals and obtain:

$$I(\lambda) = \int_{\partial \Omega'} F(z) \sin(\lambda p(z)) d\sigma(z) + O(\lambda^{-1}).$$
(7)

It is not hard to write the explicit form of smooth function F.

The following lemma describes this integral:

**Lemma 3.2.** Let *Z* be an analytic connected closed surface in  $\mathbb{R}^3$ . F(z) is a smooth function on *Z* and p(z) is analytic on *Z*. If  $J(\lambda) = \int_Z F(z)e^{i\lambda p(z)} d\sigma(z)$  does not converge to 0, as  $\lambda \to \infty$ , then  $p(z) = \text{Const}, \forall z \in Z$ .

**Proof.** Let  $\phi(u)$  be a parametrization of *Z*. We denote by *K* the set of stationary points of p(z), i.e.  $K = \{z = \phi(u): \nabla(p \circ \phi)(u) = 0\}$ . If *K* is empty then obviously  $J(\lambda)$  tends to zero as  $\lambda \to \infty$ . Moreover, if  $J(\lambda)$  does not converge to 0, then meas(K) > 0. Indeed, if it was not the case, then for any given  $\epsilon$  we could introduce such a partition of unity, that  $\limsup_{k \to \infty} |J(\lambda)| < \epsilon$ . But this cannot take place.

On the other hand  $g \equiv \nabla(p \circ \phi)$  is an analytic function and g = 0 in *K* with meas(K) > 0. Therefore (see, e.g., [14])  $g \equiv 0$ . This means that p(z) is locally constant on *Z*. But since *Z* is a connected set, then p(z) = Const,  $\forall z \in Z$ .  $\Box$ 

Now we are ready to formulate our main result (the generalization of the Kahane theorem):

**Theorem 3.3.** Let *C* be a smooth bounded, strongly convex symmetric set in  $\mathbb{R}^3$ . Let f(x) be a piecewise smooth function such, that supp  $f = \overline{\Omega}$ , where  $\Omega$  is an open set, bounded with an analytic and connected surface  $\partial \Omega$ .

If  $x_0$  is a point in  $\Omega$  where the partial sums  $S_{\lambda C} f(x_0)$  does not converge to  $f(x_0)$ , then  $\Omega = B(x_0; R)$  for some R.

**Proof.** As it immediately follows from (6), (7) and Lemma 3.2, if  $S_{\lambda C} f(x_0)$  does not converge to  $f(x_0)$ , as  $\lambda \to \infty$  then p(z) = R,  $\forall z \in \partial \Omega'$ , for some R. Therefore  $\Omega'$  has the form  $\Omega' = \{z: p(z) < R\}$ . Thus, according to our notation (4) one has  $\Omega = B(x_0; R)$ .  $\Box$ 

**Remark 3.4.** In the Pinsky and Taylor theorem we see that  $S_{\lambda C} f$  for function f with a specific supporter B(0; 1) (i.e. dual to C) may diverge at no more than one point. As it follows from Theorem 3.3 in three-dimensional case this feature is in fact inherent in a broader class of functions (i.e. functions which supporter bounded with analytic surface).

**Remark 3.5.** It is necessary to note that both conditions: analyticity and connectedness of  $\partial \Omega$  are crucial for Theorem 3.3 (see [8]).

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