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Singularities of Blaschke normal maps of convex surfaces

Singularités des applications normales de Blaschke des surfaces convexes

Kentaro Saji^a, Masaaki Umehara^b, Kotaro Yamada^c

^a Department of Mathematics, Faculty of Education, Gifu University, Yanagido 1-1, Gifu 501-1193, Japan

^b Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

^c Department of Mathematics, Tokyo Institute of Technology, O-okayama, Meguro, Tokyo 152-8551, Japan

ARTICLE INFO

Differential Geometry

Article history: Received 8 January 2010 Accepted after revision 1 April 2010 Available online 4 May 2010

Presented by Jean-Pierre Demailly

ABSTRACT

We prove that the difference between the numbers of positive swallowtails and negative swallowtails of the Blaschke normal map for a given convex surface in affine space is equal to the Euler number of the subset where the affine shape operator has negative determinant.

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RÉSUMÉ

Nous prouvons que la différence entre les nombres de queues d'aronde positives et queues d'aronde négatives de l'application normale de Blaschke, pour une surface convexe donnée dans l'espace d'affine, est égale au nombre d'Euler du sous-ensemble où l'opérateur de forme affine a un déterminant négatif.

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1. Introduction

Throughout this Note, we assume that M^2 is a compact oriented 2-manifold without boundary. Let φ be a bundle homomorphism of the tangent bundle TM^2 into a vector bundle E of rank 2 over M^2 . A point p on M^2 is called a *singular point* if the linear map $\varphi_p : T_pM \to E_p$ is not bijective. We denote by Σ_{φ} the set of singular points of φ . We assume that E is orientable, that is, there is a non-vanishing section $\mu : M^2 \to E^* \wedge E^*$, where E^* is the dual vector bundle of E. We now fix a metric \langle , \rangle on E. Multiplying a suitable C^{∞} -function on M^2 , we may assume that $\mu(e_1, e_2) = 1$ holds for any oriented orthonormal frame e_1 , e_2 on E. By using a positively oriented local coordinate system (U; u, v) of M^2 , the signed area form $d\hat{A}$, the signed area density function λ and the (*un-signed*) area form dA are defined by

 $d\hat{A} := \varphi^* \mu = \lambda \, du \wedge dv, \qquad dA := |\lambda| \, du \wedge dv.$

Both $d\hat{A}$ and dA are independent of the choice of (u, v), and are 2-forms globally defined on M^2 . When φ has no singular points, these two forms coincide up to sign. We set

$$M^+ := \big\{ p \in M^2 \setminus \Sigma_{\varphi}; \ \mathrm{d}\hat{A}_p = \mathrm{d}A_p \big\}, \qquad M^- := \big\{ p \in M^2 \setminus \Sigma_{\varphi}; \ \mathrm{d}\hat{A}_p = -\mathrm{d}A_p \big\}.$$

The singular set Σ_{φ} coincides with $\partial M^+ = \partial M^-$. A singular point $p(\in \Sigma_{\varphi})$ on M^2 is called *non-degenerate* if the derivative $d\lambda$ does not vanish at p. In a neighborhood of a non-degenerate singular point, the singular set can be parametrized as a

E-mail addresses: ksaji@gifu-u.ac.jp (K. Saji), umehara@math.sci.osaka-u.ac.jp (M. Umehara), kotaro@math.titech.ac.jp (K. Yamada).

¹⁶³¹⁻⁰⁷³X/\$ – see front matter © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. doi:10.1016/j.crma.2010.04.021

regular curve $\gamma(t)$ on M^2 , called the *singular curve*. The tangential direction of γ is called the *singular direction*. The direction of the kernel of $\varphi_{\gamma(t)}$ is called the *null direction*, which is one-dimensional. There exists a smooth non-vanishing vector field $\eta(t)$ along γ pointing in the null direction, called the *null vector field*.

Definition 1.1. Take a non-degenerate singular point $p \in M^2$ and let $\gamma(t)$ be the singular curve satisfying $\gamma(0) = p$. Then p is called an A_2 -point if the null direction $\eta(0)$ is transversal to the singular direction $\dot{\gamma}(0) = d\gamma/dt|_{t=0}$. If p is not an A_2 -point, but satisfies that $d(\dot{\gamma}(t) \land \eta(t))/dt$ does not vanish at $p = \gamma(0)$, it is called an A_3 -point, where \land is the exterior product on TM^2 . We fix an A_3 -point p. If the interior angle of the region M^- (resp. M^+) at p with respect to the pull-back metric $ds^2 := \varphi^* \langle , \rangle$ is zero, then it is called a *positive* (resp. *negative*) A_3 -point. (A_3 -points are either positive or negative, see [6]).

We now fix a metric connection *D* of (E, \langle , \rangle) . Let $\gamma(t)$ be a regular curve on M^2 consisting only of A_2 -points. Take a null vector field $\eta(t)$ such that $(\dot{\gamma}, \eta)$ is a positive frame of TM^2 along γ . Then

$$\kappa_{s}(t) = \operatorname{sgn}\left(d\lambda(\eta(t))\right) \frac{\mu(\varphi(\dot{\gamma}(t)), D_{t}\varphi(\dot{\gamma}(t)))}{\langle\varphi(\dot{\gamma}(t)), \varphi(\dot{\gamma}(t))\rangle^{3/2}}$$
(1)

is called the singular curvature of γ at t (see [5] and [6]).

For an oriented orthonormal frame field e_1 , e_2 of E defined on $U \subset M^2$, there is a unique 1-form ω on U such that $D_X e_1 = -\omega(X)e_2$, $D_X e_2 = \omega(X)e_1$. Then $d\omega$ does not depend on the choice of e_1 , e_2 , and there is a C^{∞} -function $K_{\varphi,D}$ on $M^2 \setminus \Sigma_{\varphi}$ such that

$$\mathrm{d}\omega = K_{\omega,D} \,\mathrm{d}\hat{A}.\tag{2}$$

We call $K_{\varphi,D}$ the *Gaussian curvature* of D with respect to φ . Let \overline{D} be the pull-back of D on $M^2 \setminus \Sigma_{\varphi}$. Let $\sigma(t)$ be a regular curve on $U \setminus \Sigma_{\varphi}$ with the arclength parameter t with respect to $ds^2 = \varphi^* \langle , \rangle$. We take a unit normal vector n(t) such that $(\dot{\sigma}, n)$ gives a positive frame on TM^2 . On the other hand, we take $\hat{n}(t) \in E$ such that $(\varphi(\dot{\sigma}), \hat{n})$ gives a positive frame on E. We can define two geodesic curvatures:

$$\kappa_g = \mathrm{d}s^2 (\bar{D}_t \dot{\sigma}(t), n(t)), \qquad \hat{\kappa}_g = \langle D_t \varphi (\dot{\sigma}(t)), \hat{n}(t) \rangle.$$

Here, $\hat{k}_g(t)$ is well defined even when $\sigma(t)$ passes through the set Σ_{φ} . Since $\varphi(n) = \operatorname{sgn}(\lambda)\hat{n}$, it holds that $\kappa_g = \operatorname{sgn}(\lambda)\hat{k}_g$. We set $(\bar{e}_1, \bar{e}_2) = (\varphi^{-1}(e_1), \varphi^{-1}(e_2))$ if $U \subset M^+$ and set $(\bar{e}_1, \bar{e}_2) = (\varphi^{-1}(e_2), \varphi^{-1}(e_1))$ if $U \subset M^-$. Then (\bar{e}_1, \bar{e}_2) gives an oriented orthonormal frame on TM^2 , and there is a C^{∞} -function $\theta = \theta(t)$ such that $\dot{\sigma} = \cos\theta\bar{e}_1 + \sin\theta\bar{e}_2$ and $n = -\sin\theta\bar{e}_1 + \cos\theta\bar{e}_2$. Then we get

$$\kappa_g dt = d\theta - (\operatorname{sgn} \lambda)\omega.$$
⁽³⁾

If the connection D satisfies the condition

$$D_X\varphi(Y) - D_Y\varphi(X) - \varphi([X, Y]) = 0 \tag{4}$$

for all vector fields X, Y on M^2 , $(E, \langle , \rangle, D, \varphi)$ is called a *coherent tangent bundle*. Under the condition (4), \overline{D} gives the Levi-Civita connection of ds^2 on $M^2 \setminus \Sigma_{\varphi}$, and $K_{\varphi,D}$ coincides with the usual Gaussian curvature. We consider a contractible triangular domain $\triangle ABC$ on $M^2 \setminus \Sigma_{\varphi}$ such that it lies on the left-hand side of the regular arcs AB, BC, CA which meet transversally at A, B, $C \in M^2$. By applying the Stokes formula, (2) and (3) yield that

$$\angle \mathbf{A} + \angle \mathbf{B} + \angle \mathbf{C} - \pi = \int_{\partial \triangle \mathsf{ABC}} \kappa_g \, \mathrm{d}\tau + \int_{\triangle \mathsf{ABC}} K_{\varphi, D} \, \mathrm{d}A,\tag{5}$$

where $\angle A$, $\angle B$, $\angle C$ are the interior angles of the domain $\triangle ABC$. To prove this, we do not need to assume that \overline{D} is the Levi-Civita connection. However, we must remember that $K_{\varphi,D}$ is not the usual Gaussian curvature. One crucial point in this setting is that

$$\int_{M^2} K_{\varphi,D} \, \mathrm{d}\hat{A} = \frac{1}{2\pi} \int_{M^2} \mathrm{d}\omega$$

coincides with the Euler characteristic χ_E of the vector bundle *E*. In [6] (see also [5]), the authors gave the following two Gauss–Bonnet type formulas:

$$\chi_E = \chi \left(M^+ \right) - \chi \left(M^- \right) + S_+ - S_-, \qquad 2\pi \chi \left(M^2 \right) = \int_{M^2} K_{\varphi,D} \, \mathrm{d}A + 2 \int_{\Sigma_{\varphi}} \kappa_s \, \mathrm{d}\tau, \tag{6}$$

under the assumption that $(E, \langle , \rangle, D, \varphi)$ is a coherent tangent bundle, where $d\tau$ is the arclength element on the singular set and S_+ , S_- are the numbers of positive and negative A_3 -points, respectively. After the publication of [6], the authors

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found that the proof in [6] is based only on the formula (5) and the identity $\kappa_g = \operatorname{sgn}(\lambda)\hat{\kappa}_g$. So we can conclude that the two formulas (6) hold without assuming (4). Moreover, we can generalize these two formulas to φ admitting more general singularities called "peaks"; in other words, Theorem B in [6] holds on φ without assuming (4). If $E = TM^2$, then χ_E coincides with $\chi(M^2) = \chi(M^+) + \chi(M^-)$ in our setting. So we get the following:

Theorem 1.2. Let $\varphi : TM^2 \to TM^2$ be a bundle homomorphism whose singular set consists only of A_2 and A_3 -points. Then $2\chi(M^-) = S_+ - S_-$ and $\int_{M^-} K_{\varphi,D} d\hat{A} = \int_{\Sigma_m} \kappa_s d\tau$ hold.

Let $f: M^2 \to (N^3, g)$ be an immersion into an orientable Riemannian 3-manifold. Then there is a globally defined unit normal vector field ν along f. We define the shape operator $\varphi: TM^2 \ni \nu \mapsto -D_{\nu}\nu \in TM^2$, as a bundle homomorphism, where D is the Levi-Civita connection of (N^3, g) . A singular point of φ is called an *inflection point* of f. We get the following:

Corollary 1.3 (A generalization of the Bleeker–Wilson formula). Suppose that the shape operator admits only A_2 and A_3 -points. Then $2\chi(M^-) = I_+ - I_-$ holds, where I_+ (resp. I_-) is the number of positive (resp. negative) A_3 -inflection points.

The original formula was for the case $N^3 = \mathbf{R}^3$ (see [2]). In [7], the authors pointed out that the formula holds for space forms. Also, they gave in [7] several applications of (6) under the assumption (4). However, now we can remove (4), and we get also the results that follow here.

2. Rotation of vector fields

We fix a Riemannian metric ds^2 on M^2 . There is a unique 2-form μ on M^2 such that $\mu(e_1, e_2) = 1$, where e_1, e_2 is a local oriented orthonormal frame field on M^2 . Let X be a vector field on M^2 . The C^{∞} -function $rot(X) := \mu(D_{e_1}X, D_{e_2}X)$ defined on M^2 is called *the rotation* of X, where D is the Levi-Civita connection of (M^2, ds^2) . Consider a bundle homomorphism $\varphi: TM^2 \ni v \mapsto D_v X \in TM^2$. The singular set Σ_X of φ coincides with the zeros of rot(X), called the set of *irrotational points*. Moreover, an A_3 -singular point is called an *irrotational cusp*. In fact, if $M^2 = \mathbf{R}^2$ is the Euclidean plane, then X induces a map $\tilde{X} : \mathbf{R}^2 \to \mathbf{R}^2$, and A_3 (resp. A_2) points correspond to cusps (resp. folds) of \tilde{X} (see [7]). Suppose that X admits only A_2 and A_3 -irrotational points. Then Σ_X consists of a finite disjoint union of closed regular curves $\gamma_1, \ldots, \gamma_m$ on M^2 such that M^+ lies in the left-hand side of each γ_j . Then the singular curvature on γ_j is given by $\kappa_s := \mu(\dot{X}, \ddot{X})/|\dot{X}|^3$ (we propose to call it the *irrotational curvature*), where $\dot{X} = D_{\dot{Y}_j(t)}X$ and $\ddot{X} = D_{\dot{Y}_j(t)}\dot{X}$. The following assertion follows directly from Theorem 1.2:

Proposition 2.1. Suppose that X admits only A₂ and A₃-irrotational points. Then it holds that

$$2\chi(M^{-}) = C_{+} - C_{-}, \qquad \int_{M^{-}} K_{\varphi,D} \,\mathrm{d}\hat{A} = \int_{\Sigma_{X}} \kappa_{s} \,\mathrm{d}\tau, \qquad M^{-} := \big\{ p \in M^{2}; \operatorname{rot}(X)_{p} < 0 \big\},$$

where C_+ (resp. C_-) is the number of positive (resp. negative) irrotational cusps.

3. Singularities of Blaschke normal maps on convex surfaces

Let S^2 be a 2-sphere and $f: S^2 \to \mathbb{R}^3$ a strictly convex embedding. In affine differential geometry, it is well known that there are a transversal vector field ξ along f, a torsion free connection ∇ , a bundle homomorphism $\alpha: TS^2 \to TS^2$ (called the *affine shape operator*), and a positive definite symmetric covariant tensor h such that (cf. [4]) $D_X Y = \nabla_X Y + h(X, Y)\xi$ and $D_X \xi = -\alpha(X)$ for any vector fields X, Y on S^2 , where D is the canonical affine connection on \mathbb{R}^3 . Moreover, such a structure (ξ, ∇, α, h) is uniquely determined up to a constant multiplication of ξ . Here ξ induces a map $\tilde{\xi}: S^2 \to \mathbb{R}^3$ called the *Blaschke normal map*. It is obvious that the singular points of α coincide with those of $\tilde{\xi}$.

Lemma 3.1. The Blaschke normal map $\tilde{\xi}$ is a wave front (cf. [1] for the definition of wave front).

Proof. Consider a non-zero section $L: S^2 \ni p \mapsto (\tilde{\xi}_p, v_p) \in T^* \mathbf{R}^3 = \mathbf{R}^3 \times (\mathbf{R}^3)^*$, where $v: S^2 \to (\mathbf{R}^3)^*$ is the map into the dual vector space $(\mathbf{R}^3)^*$ of \mathbf{R}^3 such that $v_p(\tilde{\xi}_p) = 1$ and $v_p(df(T_pS^2)) = \{0\}$ for each $p \in S^2$. Take a local coordinate system (u_1, u_2) of S^2 . Then we have that

$$\nu_{u_i}(f_{u_j}) = D_{\partial_i}\nu(f_{u_j}) = -\nu(D_{\partial_i}f_{u_j}) = -\nu(\nabla_{\partial_i}\partial_j + h(\partial_i, \partial_j)\tilde{\xi}) = -h(\partial_i, \partial_j) \quad (i, j = 1, 2),$$

where $\partial_i := \partial/\partial u_i$ and $f_{u_i} := df(\partial_i)$. Since *h* is positive definite, v_{u_1} , v_{u_2} are linearly independent. Moreover, v, v_{u_1} , v_{u_2} are also linearly independent, since $v(df(T_pS^2)) = 0$. In particular, *L* induces a Legendrian immersion of S^2 into the projective cotangent bundle $P(T^*\mathbf{R}^3)$ of $T^*\mathbf{R}^3$. \Box

By applying the criteria of cuspidal edges and swallowtails (cf. [7]), A_2 and A_3 -points correspond to the cuspidal edges and swallowtails of the Blaschke normal map ξ . So we get the following:

Theorem 3.2. Suppose that $\tilde{\xi}$ admits only cuspidal edges and swallowtails. Then $2\chi(M^-) = S_+ - S_-$ holds, where $M^- := \{p \in S^2; \det(\alpha(p)) < 0\}$ and S_+ (resp. S_-) is the number of positive (resp. negative) swallowtails of $\tilde{\xi}$.

A different formula for $S_+ + S_-$ is given by Izumiya and Marar [3].

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