## Differential Geometry

# Singularities of Blaschke normal maps of convex surfaces 

# Singularités des applications normales de Blaschke des surfaces convexes 

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## A R T I CLE IN F O

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#### Abstract

We prove that the difference between the numbers of positive swallowtails and negative swallowtails of the Blaschke normal map for a given convex surface in affine space is equal to the Euler number of the subset where the affine shape operator has negative determinant.


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## R É S U M É

Nous prouvons que la différence entre les nombres de queues d'aronde positives et queues d'aronde négatives de l'application normale de Blaschke, pour une surface convexe donnée dans l'espace d'affine, est égale au nombre d'Euler du sous-ensemble où l'opérateur de forme affine a un déterminant négatif.
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## 1. Introduction

Throughout this Note, we assume that $M^{2}$ is a compact oriented 2-manifold without boundary. Let $\varphi$ be a bundle homomorphism of the tangent bundle $T M^{2}$ into a vector bundle $E$ of rank 2 over $M^{2}$. A point $p$ on $M^{2}$ is called a singular point if the linear map $\varphi_{p}: T_{p} M \rightarrow E_{p}$ is not bijective. We denote by $\Sigma_{\varphi}$ the set of singular points of $\varphi$. We assume that $E$ is orientable, that is, there is a non-vanishing section $\mu: M^{2} \rightarrow E^{*} \wedge E^{*}$, where $E^{*}$ is the dual vector bundle of $E$. We now fix a metric $\langle$,$\rangle on E$. Multiplying a suitable $C^{\infty}$-function on $M^{2}$, we may assume that $\mu\left(e_{1}, e_{2}\right)=1$ holds for any oriented orthonormal frame $e_{1}, e_{2}$ on $E$. By using a positively oriented local coordinate system $(U ; u, v)$ of $M^{2}$, the signed area form $\mathrm{d} \hat{A}$, the signed area density function $\lambda$ and the (un-signed) area form $\mathrm{d} A$ are defined by

$$
\mathrm{d} \hat{A}:=\varphi^{*} \mu=\lambda \mathrm{d} u \wedge \mathrm{~d} v, \quad \mathrm{~d} A:=|\lambda| \mathrm{d} u \wedge \mathrm{~d} v
$$

Both $\mathrm{d} \hat{A}$ and $\mathrm{d} A$ are independent of the choice of $(u, v)$, and are 2 -forms globally defined on $M^{2}$. When $\varphi$ has no singular points, these two forms coincide up to sign. We set

$$
M^{+}:=\left\{p \in M^{2} \backslash \Sigma_{\varphi} ; \mathrm{d} \hat{A}_{p}=\mathrm{d} A_{p}\right\}, \quad M^{-}:=\left\{p \in M^{2} \backslash \Sigma_{\varphi} ; \mathrm{d} \hat{A}_{p}=-\mathrm{d} A_{p}\right\} .
$$

The singular set $\Sigma_{\varphi}$ coincides with $\partial M^{+}=\partial M^{-}$. A singular point $p\left(\in \Sigma_{\varphi}\right)$ on $M^{2}$ is called non-degenerate if the derivative $d \lambda$ does not vanish at $p$. In a neighborhood of a non-degenerate singular point, the singular set can be parametrized as a

[^0]regular curve $\gamma(t)$ on $M^{2}$, called the singular curve. The tangential direction of $\gamma$ is called the singular direction. The direction of the kernel of $\varphi_{\gamma(t)}$ is called the null direction, which is one-dimensional. There exists a smooth non-vanishing vector field $\eta(t)$ along $\gamma$ pointing in the null direction, called the null vector field.

Definition 1.1. Take a non-degenerate singular point $p \in M^{2}$ and let $\gamma(t)$ be the singular curve satisfying $\gamma(0)=p$. Then $p$ is called an $A_{2}$-point if the null direction $\eta(0)$ is transversal to the singular direction $\dot{\gamma}(0)=\mathrm{d} \gamma /\left.\mathrm{d} t\right|_{t=0}$. If $p$ is not an $A_{2}$-point, but satisfies that $\mathrm{d}(\dot{\gamma}(t) \wedge \eta(t)) / \mathrm{d} t$ does not vanish at $p=\gamma(0)$, it is called an $A_{3}$-point, where $\wedge$ is the exterior product on $T M^{2}$. We fix an $A_{3}$-point $p$. If the interior angle of the region $M^{-}$(resp. $M^{+}$) at $p$ with respect to the pull-back metric $\mathrm{ds}^{2}:=\varphi^{*}\langle$,$\rangle is zero, then it is called a positive (resp. negative) A_{3}$-point. ( $A_{3}$-points are either positive or negative, see [6]).

We now fix a metric connection $D$ of $(E,\langle\rangle$,$) . Let \gamma(t)$ be a regular curve on $M^{2}$ consisting only of $A_{2}$-points. Take a null vector field $\eta(t)$ such that $(\dot{\gamma}, \eta)$ is a positive frame of $T M^{2}$ along $\gamma$. Then

$$
\begin{equation*}
\kappa_{s}(t)=\operatorname{sgn}(\mathrm{d} \lambda(\eta(t))) \frac{\mu\left(\varphi(\dot{\gamma}(t)), D_{t} \varphi(\dot{\gamma}(t))\right)}{\langle\varphi(\dot{\gamma}(t)), \varphi(\dot{\gamma}(t))\rangle^{3 / 2}} \tag{1}
\end{equation*}
$$

is called the singular curvature of $\gamma$ at $t$ (see [5] and [6]).
For an oriented orthonormal frame field $e_{1}, e_{2}$ of $E$ defined on $U \subset M^{2}$, there is a unique 1 -form $\omega$ on $U$ such that $D_{X} e_{1}=-\omega(X) e_{2}, D_{X} e_{2}=\omega(X) e_{1}$. Then $d \omega$ does not depend on the choice of $e_{1}, e_{2}$, and there is a $C^{\infty}$-function $K_{\varphi, D}$ on $M^{2} \backslash \Sigma_{\varphi}$ such that

$$
\begin{equation*}
\mathrm{d} \omega=K_{\varphi, D} \mathrm{~d} \hat{A} \tag{2}
\end{equation*}
$$

We call $K_{\varphi, D}$ the Gaussian curvature of $D$ with respect to $\varphi$. Let $\bar{D}$ be the pull-back of $D$ on $M^{2} \backslash \Sigma_{\varphi}$. Let $\sigma(t)$ be a regular curve on $U \backslash \Sigma_{\varphi}$ with the arclength parameter $t$ with respect to $\mathrm{ds}^{2}=\varphi^{*}\langle$,$\rangle . We take a unit normal vector n(t)$ such that $(\dot{\sigma}, n)$ gives a positive frame on $T M^{2}$. On the other hand, we take $\hat{n}(t) \in E$ such that ( $\left.\varphi(\dot{\sigma}), \hat{n}\right)$ gives a positive frame on $E$. We can define two geodesic curvatures:

$$
\kappa_{g}=\mathrm{ds}^{2}\left(\bar{D}_{t} \dot{\sigma}(t), n(t)\right), \quad \hat{\kappa}_{g}=\left\langle D_{t} \varphi(\dot{\sigma}(t)), \hat{n}(t)\right\rangle .
$$

Here, $\hat{\kappa}_{g}(t)$ is well defined even when $\sigma(t)$ passes through the set $\Sigma_{\varphi}$. Since $\varphi(n)=\operatorname{sgn}(\lambda) \hat{n}$, it holds that $\kappa_{g}=\operatorname{sgn}(\lambda) \hat{\kappa}_{g}$. We set $\left(\bar{e}_{1}, \bar{e}_{2}\right)=\left(\varphi^{-1}\left(e_{1}\right), \varphi^{-1}\left(e_{2}\right)\right)$ if $U \subset M^{+}$and set $\left(\bar{e}_{1}, \bar{e}_{2}\right)=\left(\varphi^{-1}\left(e_{2}\right), \varphi^{-1}\left(e_{1}\right)\right)$ if $U \subset M^{-}$. Then $\left(\bar{e}_{1}, \bar{e}_{2}\right)$ gives an oriented orthonormal frame on $T M^{2}$, and there is a $C^{\infty}$-function $\theta=\theta(t)$ such that $\dot{\sigma}=\cos \theta \bar{e}_{1}+\sin \theta \bar{e}_{2}$ and $n=-\sin \theta \bar{e}_{1}+\cos \theta \bar{e}_{2}$. Then we get

$$
\begin{equation*}
\kappa_{g} \mathrm{~d} t=\mathrm{d} \theta-(\operatorname{sgn} \lambda) \omega . \tag{3}
\end{equation*}
$$

If the connection $D$ satisfies the condition

$$
\begin{equation*}
D_{X} \varphi(Y)-D_{Y} \varphi(X)-\varphi([X, Y])=0 \tag{4}
\end{equation*}
$$

for all vector fields $X, Y$ on $M^{2},(E,\langle\rangle, D,, \varphi)$ is called a coherent tangent bundle. Under the condition (4), $\bar{D}$ gives the Levi-Civita connection of ds ${ }^{2}$ on $M^{2} \backslash \Sigma_{\varphi}$, and $K_{\varphi, D}$ coincides with the usual Gaussian curvature. We consider a contractible triangular domain $\triangle \mathrm{ABC}$ on $M^{2} \backslash \Sigma_{\varphi}$ such that it lies on the left-hand side of the regular arcs $\mathrm{AB}, \mathrm{BC}, \mathrm{CA}$ which meet transversally at $A, B, C \in M^{2}$. By applying the Stokes formula, (2) and (3) yield that

$$
\begin{equation*}
\angle \mathrm{A}+\angle \mathrm{B}+\angle \mathrm{C}-\pi=\int_{\partial \triangle \mathrm{ABC}} \kappa_{g} \mathrm{~d} \tau+\int_{\triangle \mathrm{ABC}} K_{\varphi, D} \mathrm{~d} A \tag{5}
\end{equation*}
$$

where $\angle A, \angle B, \angle C$ are the interior angles of the domain $\triangle A B C$. To prove this, we do not need to assume that $\bar{D}$ is the Levi-Civita connection. However, we must remember that $K_{\varphi, D}$ is not the usual Gaussian curvature. One crucial point in this setting is that

$$
\int_{M^{2}} K_{\varphi, D} \mathrm{~d} \hat{A}=\frac{1}{2 \pi} \int_{M^{2}} \mathrm{~d} \omega
$$

coincides with the Euler characteristic $\chi_{E}$ of the vector bundle $E$. In [6] (see also [5]), the authors gave the following two Gauss-Bonnet type formulas:

$$
\begin{equation*}
\chi_{E}=\chi\left(M^{+}\right)-\chi\left(M^{-}\right)+S_{+}-S_{-}, \quad 2 \pi \chi\left(M^{2}\right)=\int_{M^{2}} K_{\varphi, D} \mathrm{~d} A+2 \int_{\Sigma_{\varphi}} \kappa_{S} \mathrm{~d} \tau \tag{6}
\end{equation*}
$$

under the assumption that $(E,\langle\rangle, D,, \varphi)$ is a coherent tangent bundle, where $\mathrm{d} \tau$ is the arclength element on the singular set and $S_{+}, S_{-}$are the numbers of positive and negative $A_{3}$-points, respectively. After the publication of [6], the authors
found that the proof in [6] is based only on the formula (5) and the identity $\kappa_{g}=\operatorname{sgn}(\lambda) \hat{\kappa}_{g}$. So we can conclude that the two formulas (6) hold without assuming (4). Moreover, we can generalize these two formulas to $\varphi$ admitting more general singularities called "peaks"; in other words, Theorem B in [6] holds on $\varphi$ without assuming (4). If $E=T M^{2}$, then $\chi_{E}$ coincides with $\chi\left(M^{2}\right)=\chi\left(M^{+}\right)+\chi\left(M^{-}\right)$in our setting. So we get the following:

Theorem 1.2. Let $\varphi: T M^{2} \rightarrow T M^{2}$ be a bundle homomorphism whose singular set consists only of $A_{2}$ and $A_{3}$-points. Then $2 \chi\left(M^{-}\right)=$ $S_{+}-S_{-}$and $\int_{M^{-}} K_{\varphi, D} \mathrm{~d} \hat{A}=\int_{\Sigma_{\varphi}} \kappa_{S} \mathrm{~d} \tau$ hold.

Let $f: M^{2} \rightarrow\left(N^{3}, g\right)$ be an immersion into an orientable Riemannian 3-manifold. Then there is a globally defined unit normal vector field $v$ along $f$. We define the shape operator $\varphi: T M^{2} \ni v \mapsto-D_{v} v \in T M^{2}$, as a bundle homomorphism, where $D$ is the Levi-Civita connection of $\left(N^{3}, g\right)$. A singular point of $\varphi$ is called an inflection point of $f$. We get the following:

Corollary 1.3 (A generalization of the Bleeker-Wilson formula). Suppose that the shape operator admits only $A_{2}$ and $A_{3}$-points. Then $2 \chi\left(M^{-}\right)=I_{+}-I_{-}$holds, where $I_{+}$(resp. $I_{-}$) is the number of positive (resp. negative) $A_{3}$-inflection points.

The original formula was for the case $N^{3}=\boldsymbol{R}^{3}$ (see [2]). In [7], the authors pointed out that the formula holds for space forms. Also, they gave in [7] several applications of (6) under the assumption (4). However, now we can remove (4), and we get also the results that follow here.

## 2. Rotation of vector fields

We fix a Riemannian metric $\mathrm{ds}^{2}$ on $M^{2}$. There is a unique 2 -form $\mu$ on $M^{2}$ such that $\mu\left(e_{1}, e_{2}\right)=1$, where $e_{1}$, $e_{2}$ is a local oriented orthonormal frame field on $M^{2}$. Let $X$ be a vector field on $M^{2}$. The $C^{\infty}$-function $\operatorname{rot}(X):=\mu\left(D_{e_{1}} X, D_{e_{2}} X\right)$ defined on $M^{2}$ is called the rotation of $X$, where $D$ is the Levi-Civita connection of $\left(M^{2}, \mathrm{ds} s^{2}\right)$. Consider a bundle homomorphism $\varphi: T M^{2} \ni v \mapsto D_{v} X \in T M^{2}$. The singular set $\Sigma_{X}$ of $\varphi$ coincides with the zeros of $\operatorname{rot}(X)$, called the set of irrotational points. Moreover, an $A_{3}$-singular point is called an irrotational cusp. In fact, if $M^{2}=\boldsymbol{R}^{2}$ is the Euclidean plane, then $X$ induces a map $\tilde{X}: \boldsymbol{R}^{2} \rightarrow \boldsymbol{R}^{2}$, and $A_{3}$ (resp. $A_{2}$ ) points correspond to cusps (resp. folds) of $\tilde{X}$ (see [7]). Suppose that $X$ admits only $A_{2}$ and $A_{3}$-irrotational points. Then $\Sigma_{X}$ consists of a finite disjoint union of closed regular curves $\gamma_{1}, \ldots, \gamma_{m}$ on $M^{2}$ such that $M^{+}$ lies in the left-hand side of each $\gamma_{j}$. Then the singular curvature on $\gamma_{j}$ is given by $\kappa_{s}:=\mu(\dot{X}, \ddot{X}) /|\dot{X}|^{3}$ (we propose to call it the irrotational curvature), where $\dot{X}=D_{\dot{\gamma}_{j}(t)} X$ and $\ddot{X}=D_{\dot{\gamma}_{j}(t)} \dot{X}$. The following assertion follows directly from Theorem 1.2:

Proposition 2.1. Suppose that $X$ admits only $A_{2}$ and $A_{3}$-irrotational points. Then it holds that

$$
2 \chi\left(M^{-}\right)=C_{+}-C_{-}, \quad \int_{M^{-}} K_{\varphi, D} \mathrm{~d} \hat{A}=\int_{\Sigma_{X}} \kappa_{S} \mathrm{~d} \tau, \quad M^{-}:=\left\{p \in M^{2} ; \operatorname{rot}(X)_{p}<0\right\},
$$

where $C_{+}$(resp. $C_{-}$) is the number of positive (resp. negative) irrotational cusps.

## 3. Singularities of Blaschke normal maps on convex surfaces

Let $S^{2}$ be a 2 -sphere and $f: S^{2} \rightarrow \boldsymbol{R}^{3}$ a strictly convex embedding. In affine differential geometry, it is well known that there are a transversal vector field $\xi$ along $f$, a torsion free connection $\nabla$, a bundle homomorphism $\alpha: T S^{2} \rightarrow T S^{2}$ (called the affine shape operator), and a positive definite symmetric covariant tensor $h$ such that (cf. [4]) $D_{X} Y=\nabla_{X} Y+h(X, Y) \xi$ and $D_{X} \xi=-\alpha(X)$ for any vector fields $X, Y$ on $S^{2}$, where $D$ is the canonical affine connection on $\boldsymbol{R}^{3}$. Moreover, such a structure $(\xi, \nabla, \alpha, h)$ is uniquely determined up to a constant multiplication of $\xi$. Here $\xi$ induces a map $\tilde{\xi}: S^{2} \rightarrow \boldsymbol{R}^{3}$ called the Blaschke normal map. It is obvious that the singular points of $\alpha$ coincide with those of $\tilde{\xi}$.

Lemma 3.1. The Blaschke normal map $\tilde{\xi}$ is a wave front (cf. [1] for the definition of wave front).
Proof. Consider a non-zero section $L: S^{2} \ni p \mapsto\left(\tilde{\xi}_{p}, v_{p}\right) \in T^{*} \boldsymbol{R}^{3}=\boldsymbol{R}^{3} \times\left(\boldsymbol{R}^{3}\right)^{*}$, where $v: S^{2} \rightarrow\left(\boldsymbol{R}^{3}\right)^{*}$ is the map into the dual vector space $\left(\boldsymbol{R}^{3}\right)^{*}$ of $\boldsymbol{R}^{3}$ such that $\nu_{p}\left(\tilde{\xi}_{p}\right)=1$ and $\nu_{p}\left(\mathrm{~d} f\left(T_{p} S^{2}\right)\right)=\{0\}$ for each $p \in S^{2}$. Take a local coordinate system ( $u_{1}, u_{2}$ ) of $S^{2}$. Then we have that

$$
v_{u_{i}}\left(f_{u_{j}}\right)=D_{\partial_{i}} v\left(f_{u_{j}}\right)=-v\left(D_{\partial_{i}} f_{u_{j}}\right)=-v\left(\nabla_{\partial_{i}} \partial_{j}+h\left(\partial_{i}, \partial_{j}\right) \tilde{\xi}\right)=-h\left(\partial_{i}, \partial_{j}\right) \quad(i, j=1,2),
$$

where $\partial_{i}:=\partial / \partial u_{i}$ and $f_{u_{i}}:=\mathrm{d} f\left(\partial_{i}\right)$. Since $h$ is positive definite, $v_{u_{1}}, v_{u_{2}}$ are linearly independent. Moreover, $v, v_{u_{1}}, v_{u_{2}}$ are also linearly independent, since $v\left(\mathrm{~d} f\left(T_{p} S^{2}\right)\right)=0$. In particular, $L$ induces a Legendrian immersion of $S^{2}$ into the projective cotangent bundle $P\left(T^{*} \boldsymbol{R}^{3}\right)$ of $T^{*} \boldsymbol{R}^{3}$.

By applying the criteria of cuspidal edges and swallowtails (cf. [7]), $A_{2}$ and $A_{3}$-points correspond to the cuspidal edges and swallowtails of the Blaschke normal map $\tilde{\xi}$. So we get the following:

Theorem 3.2. Suppose that $\tilde{\xi}$ admits only cuspidal edges and swallowtails. Then $2 \chi\left(M^{-}\right)=S_{+}-S_{-}$holds, where $M^{-}:=\{p \in$ $\left.S^{2} ; \operatorname{det}(\alpha(p))<0\right\}$ and $S_{+}\left(\right.$resp. $\left.S_{-}\right)$is the number of positive (resp. negative) swallowtails of $\tilde{\xi}$.

A different formula for $S_{+}+S_{-}$is given by Izumiya and Marar [3].

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