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Differential Geometry

Lie geometry of flat fronts in hyperbolic space

La géométrie de Lie des fronts plats dans l'éspace hyperbolique

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ABSTRACT

We propose a Lie geometric point of view on flat fronts in hyperbolic space as special Ω -surfaces and discuss the Lie geometric deformation of flat fronts.

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RÉSUMÉ

Nous proposons un point de vue de Lie géometrie sur les fronts plats dans l'éspace hyperbolique comme des surfaces Ω spéciales. Nous discutons ensuite la déformation Lie géometrique des fronts plats.

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1. Introduction

We describe (parallel families of) flat fronts in hyperbolic space in the realm of Lie sphere geometry: they turn out to be those Ω -surfaces¹ whose enveloped isothermic sphere congruences each touch a fixed sphere. This characterization is closely related to the fact that the two hyperbolic Gauss maps of a flat front are holomorphic, see [4]. The fact that "fronts" are defined by having a regular Legendre lift as well as the fact that flat fronts appear in parallel families both suggest that this is a natural viewpoint.

Since Demoulin's Ω -surfaces can be characterized as envelopes of isothermic sphere congruences – such sphere congruences always appearing in pairs that separate the curvature spheres harmonically – the rich theory of isothermic transformations becomes available. As a first example, we discuss the Calapso transformations of the isothermic sphere congruences: these induce the Lie geometric deformation of the surface as a non-rigid surface in Lie geometry, and it preserves the condition to project to a flat front in hyperbolic space – to see this we employ a pair of linear conserved quantities that the two fixed spheres give rise to.

Besides the implications for the theory of smooth flat fronts, our approach also leads to a natural integrable discretization of flat fronts in hyperbolic space, cf. [6]. Thus, we expect this exposition to be the foundation for a wealth of further research.

2. Flat fronts in hyperbolic space as Ω -surfaces

Let $f: M^2 \to H^3 = \{y \in \mathbb{R}^{3,1} \mid |y|^2 = -1, y_0 > 0\}$ be a flat front in hyperbolic space with unit normal field $t: M^2 \to S^{2,1}$. Away from umbilics and singularities, curvature line coordinates (u, v) can be introduced, $0 = t_u + \tanh \varphi f_u = t_v + \coth \varphi f_v$

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with a suitable function φ , and by the Codazzi equations (u, v) can be chosen so that the induced metric takes the form $ds^2 = \cosh^2 \varphi \, du^2 + \sinh^2 \varphi \, dv^2$ and the Gauss equation becomes harmonicity of φ . Now

$$\left(\frac{\sqrt{E}}{\sqrt{G}}\frac{\kappa_{1u}}{\kappa_1 - \kappa_2}\right)_v + \left(\frac{\sqrt{G}}{\sqrt{E}}\frac{\kappa_{2v}}{\kappa_1 - \kappa_2}\right)_u = -\varphi_{uv} + \varphi_{vu} = 0$$

characterizing f as an Ω -surface, see [2].

In order to see this geometrically, consider the orthogonal decomposition $\mathbb{R}^{4,2} = \mathbb{R}^{1,1} \oplus \mathbb{R}^{3,1}$ and fix an orthonormal basis ($\mathfrak{p},\mathfrak{q}$) of $\mathbb{R}^{1,1}$, where \mathfrak{p} defines a point sphere complex, $|\mathfrak{p}|^2 = -1$. Then

 $(u, v) \mapsto f(u, v) := \operatorname{span} \{ \mathfrak{q} + \mathfrak{f}(u, v), \mathfrak{p} + \mathfrak{t}(u, v) \}$

defines the Legendre lift of f with curvature spheres

 $s_1 = \cosh \varphi (\mathfrak{p} + \mathfrak{t}) + \sinh \varphi (\mathfrak{q} + \mathfrak{f})$ and $s_2 = \sinh \varphi (\mathfrak{p} + \mathfrak{t}) + \cosh \varphi (\mathfrak{q} + \mathfrak{f})$,

where the normalizations of the s_i have been chosen so that $s_{1u} = \varphi_u s_2$ and $s_{2v} = \varphi_v s_1$. Consequently, $s^{\pm} := s_1 \pm s_2$ are two enveloped isothermic sphere congruences for *f* since

$$s_{uv}^{\pm} = (\pm \varphi_{uv} + \varphi_u \varphi_v) s^{\pm}$$

again characterizing the surface as an Ω -surface, see [2] and [3]. Note that the spheres $s^{\pm}(u, v)$ separate the curvature spheres harmonically on the projective line (contact element) given by f(u, v).

As the Legendre map of a flat front is required to be regular this second analysis and, in particular, the isothermic sphere congruences s^{\pm} extend through the singularities of f.

To characterize those Ω -surfaces that project to flat fronts in hyperbolic space, first note that the second envelopes of the two isothermic sphere congruences s^{\pm} are the hyperbolic Gauss maps of the flat front: $s^{\pm} \perp \mathfrak{p} \pm \mathfrak{q} =: q^{\pm}$, that is, s^{\pm} has oriented contact with the infinity sphere of hyperbolic space q^{\pm} , equipped with opposite orientations.

Thus, suppose $f = \text{span}\{s^+, s^-\}$ is an Ω -surface, given in terms of a pair of (isothermic) sphere congruences s^{\pm} that separate the curvature spheres harmonically. Assume that each of the sphere congruences s^{\pm} envelops a fixed sphere q^{\pm} so that q^+ and q^- do not span a contact element, that is, $(q^+, q^-) \neq 0$; wlog., $(q^+, q^-) = -2$. Next we fix a point sphere complex $\mathfrak{p} := \frac{1}{2}(q^+ + q^-)$ and assume that both of the spheres s^{\pm} never become point spheres, $(s^{\pm}, \mathfrak{p}) \neq 0$. Then

$$\mathfrak{f} := -\left(\frac{q^+}{2} + \frac{s^+}{(s^+, q^-)}\right) + \left(\frac{q^-}{2} + \frac{s^-}{(s^-, q^+)}\right) \quad \text{and} \quad \mathfrak{t} := -\left(\frac{q^+}{2} + \frac{s^+}{(s^+, q^-)}\right) - \left(\frac{q^-}{2} + \frac{s^-}{(s^-, q^+)}\right)$$

take values in $\mathbb{R}^{3,1} = \{q^+, q^-\}^{\perp}$; moreover, $|\mathfrak{f}|^2 = -1$ so that \mathfrak{f} maps into one of the hyperbolic spaces with infinity sphere q^{\pm} . Using the contact condition $(ds^+, s^-) = 0$ it is straightforward to show that \mathfrak{t} is a unit normal field of \mathfrak{f} . Finally, to see that f is a flat front, we wheel out that s^{\pm} separate the curvature spheres $s_i = -(1+\kappa_i)\frac{s^+}{(s^+,a^-)} - (1-\kappa_i)\frac{s^-}{(s^-,a^+)}$ harmonically, that is, the cross ratio $\frac{\kappa_1-1}{\kappa_1+1}\frac{\kappa_2+1}{\kappa_2-1} = -1$ where f immerses, implying $\kappa_1\kappa_2 = 1$. Note that a different normalization $\tilde{q}^{\pm} = e^{\pm t}q^{\pm}$ of the q^{\pm} leads to a parallel flat front.

Thus we have proved²: Flat fronts in hyperbolic space are those Ω -surfaces with a pair of isothermic sphere congruences that each touch a fixed sphere, where the fixed spheres do not span a contact element. The two fixed spheres yield the two orientations of the infinity sphere of the hyperbolic ambient space of the flat front; this determines the point sphere complex, hence the flat front, up to parallel transformation.

3. Deformation of flat fronts in hyperbolic space

Demoulin [2] introduced Ω -surfaces as the Lie geometric analogue of the *R*-surfaces of projective geometry. It is therefore not too surprising that Ω -surfaces are the generic³ deformable surfaces of Lie geometry, see [1, §85] or [9]. Indeed, each of the isothermic sphere congruences s^{\pm} enveloped by an Ω -surface f comes with its Calapso transformations $T^{\pm}(\lambda): M^2 \to O(4, 2)$, where $dT^{\pm}(\lambda) = T^{\pm}(\lambda)\lambda\tau^{\pm}$ with $\tau^{\pm} = s^{\pm} \wedge \star ds^{\pm}$ for Moutard lifts s^{\pm} of the isothermic sphere congruences, $s^{\pm} \wedge s^{\pm}_{uv} = 0$, and the Hodge- \star operator of Fubini's quadratic form, $\star du = du$ and $\star dv = -dv$. Aligning the Moutard lifts to reflect across the Lie cyclides,⁴ $s^{\pm} = s_1 \pm s_2$ for suitable lifts of the curvature spheres,

$$\tau^+ + \frac{1}{2}d(s^+ \wedge s^-) = \tau^- + \frac{1}{2}d(s^- \wedge s^+) =: \tau$$

² Observe that we did not use that the harmonically separating sphere congruences s^{\pm} are isothermic: indeed an immersed sphere congruence that touches a fixed sphere is automatically Ribaucour - hence a Legendre map with two enveloped sphere congruences that separate the curvature spheres harmonically and each touches a fixed sphere is automatically an Ω -surface, cf. [1, §85].

 $^{^{3}}$ The other, non-generic class of deformable surfaces, corresponding to the R_{0} -surfaces, being those where one of the curvature sphere congruences becomes isothermic in the sense that it has a Moutard lift, e.g., channel surfaces.

⁴ This is always possible; we have seen it above in the flat front case that we will discuss.

since $s_1 \wedge s_{2v} = s_2 \wedge s_{1u} = 0$, which leads to

$$T^{+}(\lambda)\left(1+\frac{\lambda}{2}s^{+}\wedge s^{-}\right)=T^{-}(\lambda)\left(1+\frac{\lambda}{2}s^{-}\wedge s^{+}\right)=T(\lambda),$$

where $T(\lambda)$ yields the Lie-geometric deformation of $f = \text{span}\{s^+, s^-\}$ via $f(\lambda) = T(\lambda)f$, see [9]. Observe that, as f is a congruence of null 2-planes, T and T^{\pm} act the same on s^+ and s^- , hence on f: The Calapso transformations T^{\pm} of the enveloped isothermic sphere congruences s^{\pm} of an Ω -surface f yield the Lie geometric deformation $\lambda \to T(\lambda)f = T^{\pm}(\lambda)f$ of f as a Lie deformable surface.

In the flat front case $s^{\pm} = e^{\pm \varphi} (\mathfrak{p} \pm \mathfrak{q} + \mathfrak{t} \pm \mathfrak{f})$ and $ds^{\pm} = \pm d\varphi s^{\pm} - e^{\pm \varphi} \star ds^{\pm}$ so that

$$\tau = -\frac{1}{2} \{ s^+ \wedge ds^- + s^- \wedge ds^+ \} - d\varphi \, s^+ \wedge s^- = -(\mathfrak{p} + \mathfrak{t}) \wedge d\mathfrak{t} + (\mathfrak{q} + \mathfrak{f}) \wedge d\mathfrak{f}$$

and the two fixed sphere congruences q^{\pm} give rise to two linear conserved quantities

$$p^{\pm}(\lambda) = \left(1 + \frac{\lambda}{2}s^{\pm} \wedge s^{\mp}\right)^{-1}q^{\pm} = q^{\pm} + \frac{\lambda}{2}(q^{\pm}, s^{\mp})s^{\pm} = (1 - \lambda)(\mathfrak{p} \pm \mathfrak{q}) - \lambda(\mathfrak{t} \pm \mathfrak{f})$$

for the Lie geometric deformation $T(\lambda)$ since $d(Tp^{\pm})(\lambda) = T(\lambda)(d + \lambda\tau)p^{\pm}(\lambda) = 0$. Hence, the deformed surfaces are flat fronts in hyperbolic space as long as $(p^+, p^-)(\lambda) = -2(1 - 2\lambda) \neq 0$ with $(Tp^{\pm})(\lambda)$ as the infinity sphere of the hyperbolic ambient space with its two orientations: *The Lie geometric deformation of a flat front in hyperbolic space yields a 1-parameter family of flat fronts in hyperbolic space.*

Normalizing the conserved quantities, $p^{\pm}(\lambda) \rightarrow \frac{p^{\pm}(\lambda)}{\sqrt{1-2\lambda}}$, and following the construction in the previous section we obtain

$$f(\lambda) = T(\lambda) \frac{h^+(\lambda) - h^-(\lambda)}{2\sqrt{1 - 2\lambda}} \text{ and } t(\lambda) = T(\lambda) \frac{h^+(\lambda) + h^-(\lambda)}{2\sqrt{1 - 2\lambda}}$$

with $h^{\pm}(\lambda) = -\lambda(\mathfrak{p} \pm \mathfrak{q}) + (1 - \lambda)(\mathfrak{t} \pm \mathfrak{f})$. Choosing constants of integration so that $Tp^{\pm} \parallel q^{\pm}$ for every $\lambda \neq \frac{1}{2}$, the deformation is confined to $\mathbb{R}^{3,1} = \{q^+, q^-\}^{\perp}$. With $\mathfrak{e}_1 := \pm e^{\pm\varphi}(\mathfrak{t} \pm \mathfrak{f})_u$ and $\mathfrak{e}_2 := e^{\pm\varphi}(\mathfrak{t} \pm \mathfrak{f})_v$ we obtain frames $F(\lambda) := T(\lambda)(\mathfrak{e}_1, \mathfrak{e}_2, \frac{h^+(\lambda)}{\sqrt{1-2\lambda}}, \frac{h^-(\lambda)}{\sqrt{1-2\lambda}})$ for the family of point-pair maps into the conformal 2-sphere given by the two hyperbolic Gauss maps of the $\mathfrak{f}(\lambda)$. As (degenerate) Darboux pairs these are curved flats in the space of point pairs, see [5, §5.5.20 or Sect. 8.7]. Now

$$(d + \lambda\tau)h^{\pm}(\lambda) = \sqrt{1 - 2\lambda}e^{\mp\varphi}(\pm \mathfrak{e}_{1} du + \mathfrak{e}_{2} dv),$$

$$(d + \lambda\tau)\mathfrak{e}_{1} = (\varphi_{v} du - \varphi_{u} dv)\mathfrak{e}_{2} + \frac{\sqrt{1 - 2\lambda}}{2} (e^{\varphi}h^{+}(\lambda) - e^{-\varphi}h^{-}(\lambda)) du,$$

$$(d + \lambda\tau)\mathfrak{e}_{2} = (-\varphi_{v} du + \varphi_{u} dv)\mathfrak{e}_{1} - \frac{\sqrt{1 - 2\lambda}}{2} (e^{\varphi}h^{+}(\lambda) + e^{-\varphi}h^{-}(\lambda)) dv,$$

showing that $T(\lambda)(\frac{h^+(\lambda)}{\sqrt{1-2\lambda}}, \frac{h^-(\lambda)}{\sqrt{1-2\lambda}})$ is the 1-parameter family of Darboux pairs obtained from the curved flat associated family [5, §3.3.3]: The Lie geometric deformation of a flat front in hyperbolic space with parameter λ yields the curved flat associated family with parameter⁵ $\sqrt{1-2\lambda}$ of the Darboux pair of its hyperbolic Gauss maps in the symmetric space of point pairs in S².

Note that a Darboux pair in S^2 gives rise to a unique parallel family of flat fronts in hyperbolic space by determining the orthogonal surfaces of the cyclic system given by the circles intersecting the infinity sphere of hyperbolic space orthogonally in the points of the hyperbolic Gauss maps,⁶ see [7]. Thus the family of Darboux pairs is sufficient to determine the corresponding Lie geometric deformation of (parallel families of) flat fronts.

In the case of the peach front, see [8], the curved flat system can explicitly be integrated: when $\sqrt{1-2\lambda}$ is real, the $f(\lambda)$ become the snowman type flat fronts, see [7], or a new type of flat fronts whereas, when the parameter is imaginary, we obtain the hourglass type flat fronts, see [7].

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⁵ Note that an imaginary parameter still yields a real geometry, as the semi-Riemannian symmetric space of point pairs in a conformal *n*-sphere is self dual; a change of sign of the curved flat parameter does not change the geometry of the curved flat, see [5, §5.5.19].

⁶ This, in fact, shows that flat fronts are Guichard surfaces, which were shown to be Ω -surfaces in [2].

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