Topology

# The negative answer to Kameko's conjecture on the hit problem 

# Un contre-exemple à la conjecture de Kameko 

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## A R T I C L E IN F O

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#### Abstract

This Note gives a counter-example to Kameko's conjecture, stating an explicit upper bound for the cardinal of a minimal system of generators - as module over the Steenrod's algebra $\mathcal{A}$ - of the polynomial algebra $P_{k}$ in $k$ generators (of degree 1 ) over the field $\mathbb{F}_{2}$. The conjecture is true for $k=3$ (Kameko (1990) [6]), $k=4$ (Kameko (2003) [7] and the author of this Note; Sum (preprint) [15]), but false for $k>4$. In order to give the counterexample we restrict to some degrees and prove a recurrence relation for the cardinal of a minimal system of generators in these degrees. It results as an easy consequence that the conjecture is false for $k>4$.


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## R É S U M É

Cette Note donne un contre-exemple à la conjecture de Kameko. Celle ci donnait une borne supérieure explicite pour le cardinal d'un système minimal de générateurs - comme module sur l'algèbre de Steenrod $\mathcal{A}$ - de l'algèbre polynomiale $P_{k}$ en $k$ générateurs (de degré 1 ) sur le corps $\mathbb{F}_{2}$. La conjecture est vraie pour $k=3$ (Kameko, thèse Johns Hopkins University, 1990), récemment démontrée par Kameko, Nam et l'auteur de la Note pour $k=4$, qui montre ici qu'elle est fausse pour $k>4$. Pour donner ce contre-exemple l'auteur se restreint à certains degrés, et démontre une formule de récurrence pour le cardinal d'un système minimal de générateurs en ces degrés. C'est alors une conséquence facile de cette formule qui montre que la conjecture est fausse si $k>4$.
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Let $V_{k}$ be an elementary abelian 2-group of rank $k$. Denote by $B V_{k}$ the classifying space of $V_{k}$. Then

$$
P_{k}:=H^{*}\left(B V_{k}\right) \cong \mathbb{F}_{2}\left[x_{1}, x_{2}, \ldots, x_{k}\right],
$$

a polynomial algebra on $k$ generators $x_{1}, x_{2}, \ldots, x_{k}$, each of degree 1 . Here the cohomology is taken with coefficients in the prime field $\mathbb{F}_{2}$ of two elements. Being the cohomology of a space, $P_{k}$ is a module over the mod 2 Steenrod algebra $\mathcal{A}$.

A polynomial $f$ in $P_{k}$ is called hit if it can be written as a finite sum $f=\sum_{i>0} S q^{i}\left(f_{i}\right)$ for some polynomials $f_{i}$. That means $f$ belongs to $\mathcal{A}^{+} P_{k}$, where $\mathcal{A}^{+}$denotes the augmentation ideal in $\mathcal{A}$. We are interested in the hit problem, set up by F. Peterson, of finding a minimal set of generators for the polynomial algebra $P_{k}$ as a module over the Steenrod algebra. In other words, we want to find a basis of the $\mathbb{F}_{2}$-vector space $\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}$.

[^0]The general linear group $G L_{k}=G L_{k}\left(\mathbb{F}_{2}\right)$ acts naturally on $P_{k}$ by matrix substitution. Since the two actions of $\mathcal{A}$ and $G L_{k}$ upon $P_{k}$ commute with each other, there is an action of $G L_{k}$ on $\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}$. The subspace of degree $n$ homogeneous polynomials $\left(P_{k}\right)_{n}$ and its quotient $\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{n}$ are $G L_{k}$-subspaces of the spaces $P_{k}$ and $\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}$ respectively.

The hit problem was first studied by Peterson [11], Wood [16], Singer [14], and Priddy [12], who showed its relationship to several classical problems respectively in cobordism theory, modular representation theory, Adams spectral sequence for the stable homotopy of spheres, and stable homotopy type of classifying spaces of finite groups. The tensor product $\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}$ was explicitly calculated by Peterson [11] for $k=1,2$, by Kameko [6] for $k=3$, and recently by Kameko [7], Nam [10] and by us [15] for $k=4$.

Several aspects of the hit problem were then investigated by many authors (e.g. Peterson, Wood, Singer, Kameko, Hubbuck, Hu'ng, Nam and others).

Carlisle and Wood showed in [3] that the dimension of the vector space $\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{n}$ is uniformly bounded by a number depended only on $k$. In 1990, Kameko made the following conjecture in his Johns Hopkins University Ph.D. thesis [6]:

Conjecture 1. (See Kameko [6].) For every nonnegative integer n,

$$
\operatorname{dim}\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{n} \leqslant \prod_{1 \leqslant i \leqslant k}\left(2^{i}-1\right)
$$

The conjecture was shown by Kameko himself for $k \leqslant 3$ in [6], and recently proved by Kameko [7], Nam [10] and us [15] for $k=4$. The purpose of this Note is to give a negative answer to this conjecture for any $k>4$. The following is the main result of the Note:

Theorem 2. Kameko's conjecture is not true for any $k>4$.

One of our main tools in the proof of this theorem is the so-called Kameko squaring operation

$$
S q^{0}: \mathbb{F}_{2} \underset{G L_{k}}{\otimes} P H_{*}\left(B V_{k}\right) \rightarrow \mathbb{F}_{2} \underset{G L_{k}}{\otimes} P H_{*}\left(B V_{k}\right) .
$$

Here $H_{*}\left(B V_{k}\right)$ is homology with $\mathbb{F}_{2}$ coefficients, and $P H_{*}\left(B V_{k}\right)$ denotes the primitive subspace consisting of all elements in the space $H_{*}\left(B V_{k}\right)$, which are annihilated by every positive-degree operation in the mod 2 Steenrod algebra; therefore, $\mathbb{F}_{2} \otimes_{G L_{k}} P H_{*}\left(B V_{k}\right)$ is dual to $\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)^{G L_{k}}$. It was recognized by Boardman [1] for $k=3$ and Minami [8] for general $k$ that the Kameko squaring operation commutes with the classical squaring operation on the cohomology of the Steenrod algebra through the Singer algebraic transfer

$$
\operatorname{Tr}_{k}: \mathbb{F}_{2} \underset{G L_{k}}{\otimes} P H_{d}\left(B V_{k}\right) \rightarrow \operatorname{Ext}_{\mathcal{A}}^{k, k+d}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)
$$

Boardman used this fact to show that $\mathrm{Tr}_{3}$ is an isomorphism. Bruner, Hà and Hu'ng [2] applied it to prove that $\mathrm{Tr}_{4}$ does not detect any element in the usual family $\left\{g_{i}\right\}_{i>0}$ of $\operatorname{Ext}_{\mathcal{A}}^{4}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)$, and therefore gave a negative answer to Minami's conjecture predicting that the algebraic transfer becomes an isomorphism after localizing by inverting the squaring operation $S q^{0}$. Further, Hu'ng showed in [5] that the Kameko squaring operation is eventually isomorphic, and exploited this fact to prove that $\mathrm{Tr}_{k}$ is not an isomorphism in infinitely many degrees for any $k \geqslant 4$. Recently, Hu'ng and his collaborators have completely determined the image of the fourth transfer $\operatorname{Tr}_{4}$.

The $\mu$-function is one of the numerical functions that have much been used in the context of the hit problem. For a positive integer $n$, by $\mu(n)$ one means the smallest number $r$ for which it is possible to write $n=\sum_{1 \leqslant i \leqslant r}\left(2^{d_{i}}-1\right)$, where $d_{i}>0$. Peterson [11] made the following conjecture, which was subsequently proved by Wood [16]:

Theorem $3\left(\right.$ Wood [16]). If $\mu(n)>k$ then $\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{n}=0$.
The dual of the Kameko squaring $S q_{*}^{0}:\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)^{G L_{k}} \rightarrow\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)^{G L_{k}}$ is induced by the $G L_{k}$-homomorphism $\widetilde{\mathcal{S q}_{*}^{0}}$ : $\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k} \rightarrow \mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}$. The latter is induced by the $\mathcal{A}$-linear map, also denoted by $\tilde{S q}_{*}^{0}: P_{k} \rightarrow P_{k}$, given by

$$
\tilde{S q}_{*}^{0}(x)= \begin{cases}y, & \text { if } x=x_{1} x_{2} \ldots x_{k} y^{2} \\ 0, & \text { otherwise }\end{cases}
$$

for any monomial $x \in P_{k}$.
Observe that obviously $\tilde{S q}_{*}^{0}$ is surjective on $P_{k}$ and therefore on $\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}$. So, one gets

$$
\operatorname{dim}\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{2 m+k}=\operatorname{dim} \operatorname{Ker}\left(\widetilde{S q}_{*}^{0}\right)_{m}^{k}+\operatorname{dim}\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{m}
$$

for any positive integer $m$. Here $\left(\widetilde{S q}_{*}^{0}\right)_{m}^{k}:\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{2 m+k} \rightarrow\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{m}$ denotes the squaring $\widetilde{S q_{*}^{0}}$ in degree $2 m+k$.

Theorem 4 (Kameko [6]). Let $m$ be a positive integer. If $\mu(2 m+k)=k$ then $\left(\widetilde{S q}_{*}^{0}\right)_{m}^{k}:\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{2 m+k} \rightarrow\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{m}$ is an isomorphism of $G L_{k}$-modules.

Based on Theorems 3, 4 and the fact that if $\mu(2 m+k)=k$ then $\mu(m) \leqslant k$ (see [6]), the hit problem is reduced to the case of degree $n$ with $\mu(n)<k$. A routine computation shows that if $\mu(n)=s$ then $n$ can be written in the form

$$
n=f\left(d_{1}, d_{2}, \ldots, d_{s}\right):=2^{d_{1}}+2^{d_{2}}+\cdots+2^{d_{s-1}}+2^{d_{s}}-s
$$

where $d_{1}>d_{2}>\cdots>d_{s-1} \geqslant d_{s}>0$.
The hit problem in the case of degree $n$ of this form with $s=k-1, d_{i-1}-d_{i}>1$ for $2 \leqslant i<k$ and $d_{k-1}>1$ was studied by Crabb and Hubbuck [4], Nam [9] and Repka and Selick [13]. Nam [9] showed that in this case if $d_{i-1}-d_{i} \geqslant i$ for $2 \leqslant i<k$ and $d_{k-1} \geqslant k$ then Kameko's conjecture holds.

In order to prove the main theorem, we study the hit problem in the case where the degree $n$ is of the above form with $s=k-1$ and $d_{k-2}=d_{k-1}$. From a result in Nam [9], we get the following which gives an inductive formula for the dimension of $\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{n}$ in this case:

Theorem 5 (Nam [9]). Let $n=f\left(d_{1}, d_{2}, \ldots, d_{k-2}, d_{k-2}\right)$ with $d_{i}$ positive integers such that $d_{1}>d_{2}>\cdots>d_{k-3}>d_{k-2}$, and let $m=f\left(d_{1}-d_{k-2}, d_{2}-d_{k-2}, \ldots, d_{k-3}-d_{k-2}\right)$. If $d_{k-2} \geqslant k \geqslant 4$, then

$$
\operatorname{dim}\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{n}=\left(2^{k}-1\right) \operatorname{dim}\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k-1}\right)_{m}
$$

Observe that

$$
f\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k-3}\right)=2 f\left(\lambda_{1}-1, \lambda_{2}-1, \ldots, \lambda_{k-4}-1, \lambda_{k-3}-2, \lambda_{k-3}-2\right)+k-1,
$$

where $\lambda_{i}=d_{i}-d_{k-2}$ for $i=1,2, \ldots, k-3$. So, by induction on $k$ using Theorem 5 and the fact that the squaring $\widetilde{S q}_{*}^{0}$ is surjective on $\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}$, we have

Theorem 6. Let $\ell_{1}, \ell_{2}, \ldots, \ell_{k-1}$ be integers such that $\ell_{1}>\ell_{2}>\cdots>\ell_{k-1} \geqslant 0$. Set $n_{r}=f\left(\ell_{1}-\ell_{r-1}, \ldots, \ell_{r-3}-\ell_{r-1}\right.$, $\left.\ell_{r-2}-\ell_{r-1}-1, \ell_{r-2}-\ell_{r-1}-1\right)$ with $r=3,4, \ldots, k$. If $\ell_{i-2}-\ell_{i-1}>i$ for $3 \leqslant i \leqslant k$ and $k \geqslant 5$, then

$$
\operatorname{dim}\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{2 n_{k}+k}=\prod_{1 \leqslant i \leqslant k}\left(2^{i}-1\right)+\sum_{5 \leqslant r \leqslant k}\left(\prod_{r+1 \leqslant i \leqslant k}\left(2^{i}-1\right)\right) \operatorname{dim} \operatorname{Ker}\left(\tilde{S q}_{*}^{0}\right)_{n_{r}}^{r}
$$

Here, by convention, $\prod_{r+1 \leqslant i \leqslant k}\left(2^{i}-1\right)=1$ for $r=k$.
In order to conclude that Kameko's conjecture is false in degree $2 n_{k}+k$ for any $k \geqslant 5$, it suffices to show that $\operatorname{Ker}\left(\widetilde{S q_{*}}\right)_{n_{r}}^{r}$ is nonzero. Moreover, from Theorem 5 and the inductive scheme, it is sufficient to show that the conjecture fails for $k=5$.

Consider the element $x=x_{1}^{2^{e_{1}}-1} x_{2}^{2^{e_{2}}-1} x_{3}^{2_{3}-1}$ in degree $2 n_{5}+5=f\left(e_{1}, e_{2}, e_{3}\right)$, where $e_{i}=d_{i}-d_{4}+1$ for $i=1,2,3$. The element is called a spike, i.e. a monomial whose exponents are all of the form $2^{e}-1$ for some $e$.

It is well known that the class $[x]$ in $\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{5}\right)_{2 n_{5}+5}$ of a spike $x$ is nonzero. Indeed, one has

$$
S q^{a} x_{j}^{2^{e}-1-a}=\binom{2^{e}-1-a}{a} x_{j}^{2^{e}-1}=0
$$

as $\binom{2^{e}-1-a}{a}=0$ for arbitrary $j$ and any $a>0$. Hence, a spike cannot be hit by any Steenrod operation of positive degree.
On the other hand, since the exponents of $x_{4}$ and $x_{5}$ in $x$ are zero, $\left(\widetilde{S q}_{*}^{0}\right)_{n_{5}}^{5}([x])=0$. Thus, we have $\operatorname{Ker}\left(\widetilde{S q}_{*}^{0}\right)_{n_{5}}^{5} \neq 0$. This completes the proof of Theorem 2.

However, we have $\operatorname{dim}\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{2 n_{k}+k}=\operatorname{dim}\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{n_{k}}+\operatorname{dim} \operatorname{Ker}\left(\tilde{S q_{*}}\right)_{n_{k}}^{r}$. Then, by Theorem 6,

$$
\operatorname{dim}\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{n_{k}}=\prod_{1 \leqslant i \leqslant k}\left(2^{i}-1\right)+\sum_{5 \leqslant r<k}\left(\prod_{r+1 \leqslant i \leqslant k}\left(2^{i}-1\right)\right) \operatorname{dim} \operatorname{Ker}\left(\widetilde{S q}_{*}^{0}\right)_{n_{r}}^{r}
$$

Hence, for $k>5$, Kameko's conjecture is not true in degree $n_{k}$.
The following gives our prediction on the dimension of $\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{2 n_{k}+k}$ and $\left(\mathbb{F}_{2} \otimes_{\mathcal{A}} P_{k}\right)_{n_{k}}$ in the appropriate case:
Conjecture 7. Under the hypotheses of Theorem 6,

$$
\operatorname{dim} \operatorname{Ker}\left(\widetilde{S q}_{*}^{0}\right)_{n_{k}}^{k}=\prod_{3 \leqslant i \leqslant k}\left(2^{i}-1\right)
$$

The proofs of the results of this Note will be published in detail elsewhere.

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