The negative answer to Kameko's conjecture on the hit problem

Un contre-exemple à la conjecture de Kameko

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**Abstract**

This Note gives a counter-example to Kameko's conjecture, stating an explicit upper bound for the cardinal of a minimal system of generators – as module over the Steenrod's algebra $A$ – of the polynomial algebra $P_k$ in $k$ generators (of degree 1) over the field $F_2$. The conjecture is true for $k = 3$ (Kameko (1990) [6]), $k = 4$ (Kameko (2003) [7] and the author of this Note: Sum (preprint) [15]), but false for $k > 4$. In order to give the counter-example we restrict to some degrees and prove a recurrence relation for the cardinal of a minimal system of generators in these degrees. It results as an easy consequence that the conjecture is false for $k > 4$.

**Résumé**

Cette Note donne un contre-exemple à la conjecture de Kameko. Celle ci donnait une borne supérieure explicite pour le cardinal d'un système minimal de générateurs – comme module sur l'algèbre de Steenrod $A$ – de l'algèbre polynomiale $P_k$ en $k$ générateurs (de degré 1) sur le corps $F_2$. La conjecture est vraie pour $k = 3$ (Kameko, thèse Johns Hopkins University, 1990), récemment démontrée par Kameko, Nam et l'auteur de la Note pour $k = 4$, qui montre ici qu'elle est fausse pour $k > 4$. Pour donner ce contre-exemple l'auteur se restreint à certains degrés, et démontre une formule de récurrence pour le cardinal d'un système minimal de générateurs en ces degrés. C'est alors une conséquence facile de cette formule qui montre que la conjecture est fausse si $k > 4$.

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Let $V_k$ be an elementary abelian 2-group of rank $k$. Denote by $BV_k$ the classifying space of $V_k$. Then

$$P_k := H^*(BV_k) \cong F_2[x_1, x_2, \ldots, x_k],$$

a polynomial algebra on $k$ generators $x_1, x_2, \ldots, x_k$, each of degree 1. Here the cohomology is taken with coefficients in the prime field $F_2$ of two elements. Being the cohomology of a space, $P_k$ is a module over the mod 2 Steenrod algebra $A$.

A polynomial $f$ in $P_k$ is called hit if it can be written as a finite sum $f = \sum_{i \geq 0} Sq^i(f_i)$ for some polynomials $f_i$. That means $f$ belongs to $A^+ P_k$, where $A^+$ denotes the augmentation ideal in $A$. We are interested in the hit problem, set up by F. Peterson, of finding a minimal set of generators for the polynomial algebra $P_k$ as a module over the Steenrod algebra. In other words, we want to find a basis of the $F_2$-vector space $F_2 \otimes_A P_k$.

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The general linear group \( GL_k = GL_k(\mathbb{F}_2) \) acts naturally on \( P_k \) by matrix substitution. Since the two actions of \( \mathcal{A} \) and \( GL_k \) upon \( P_k \) commute with each other, there is an action of \( GL_k \) on \( \mathbb{F}_2 \otimes_{\mathcal{A}} P_k \). The subspace of degree \( n \) homogeneous polynomials \( (P_k)_n \) and its quotient \( (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_n \) are \( GL_k \)-subspaces of the spaces \( P_k \) and \( \mathbb{F}_2 \otimes_{\mathcal{A}} P_k \) respectively.

The hit problem was first studied by Peterson [11], Wood [16], Singer [14], and Priddy [12], who showed its relationship to several classical problems respectively in cobordism theory, modular representation theory, Adams spectral sequence polynomials \( GL_k \), and its quotient \( (\mathbb{F}_2 \otimes_{\mathcal{A}} P_k)_n \) are \( GL_k \)-subspaces of the spaces \( P_k \) and \( \mathbb{F}_2 \otimes_{\mathcal{A}} P_k \) respectively.

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Theorem 4 (Kameko [6]). Let \( m \) be a positive integer. If \( \mu(2m + k) = k \) then \((\tilde{S}_{\lambda}^0)_{nk}^k : (F_2 \otimes_A P_k)_{2m+k} \rightarrow (F_2 \otimes_A P_k)_m\) is an isomorphism of \( GL_k \)-modules.

Based on Theorems 3 and 4 and the fact that if \( \mu(2m + k) = k \) then \( \mu(m) \leq k \) (see [6]), the hit problem is reduced to the case of degree \( n \) with \( \mu(n) < k \). A routine computation shows that if \( \mu(n) = s \) then \( n \) can be written in the form

\[
n = f(d_1, d_2, \ldots, d_s) := 2^{d_1} + 2^{d_2} + \cdots + 2^{d_{s-1}} + 2^{d_s} - s,
\]

where \( d_1 > d_2 > \cdots > d_{s-1} \geq d_s > 0 \).

The hit problem in the case of degree \( n \) of this form with \( s = k - 1, d_{i+1} > d_i > 1 \) for \( 2 \leq i < k \) and \( d_{k-1} > 1 \) was studied by Crabb and Hubbuck [4], Nam [9] and Repka and Selick [13]. Nam [9] showed that in this case if \( d_{i-1} - d_i \geq i \) for \( 2 \leq i < k \) and \( d_{k-1} > k \) then Kameko's conjecture holds.

In order to prove the main theorem, we study the hit problem in the case where the degree \( n \) is of the above form with \( s = k - 1 \) and \( d_{k-2} = d_{k-1} \). From a result in Nam [9], we get the following which gives an inductive formula for the dimension of \((F_2 \otimes_A P_k)_n\) in this case:

**Theorem 5 (Nam [9]).** Let \( n = f(d_1, d_2, \ldots, d_{k-2}, d_{k-1}) \) with \( d_i \) positive integers such that \( d_1 > d_2 > \cdots > d_{k-3} > d_{k-2} \), and let \( m = f(d_1 - d_{k-2}, d_2 - d_{k-2}, \ldots , d_{k-3} - d_{k-2}) \). If \( d_{k-2} \geq k \geq 4 \), then

\[
\dim (F_2 \otimes_A P_k)_n = (2^k - 1) \dim (F_2 \otimes_A P_{k-1})_m.
\]

Observe that

\[
f(\lambda_1, \lambda_2, \ldots, \lambda_{k-3}) = 2f(\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_{k-4} - 1, \lambda_{k-3} - 2, \lambda_{k-3} - 2) + k - 1,
\]

where \( \lambda_i = d_i - d_{k-2} \) for \( i = 1, 2, \ldots, k - 3 \). So, by induction on \( k \) using Theorem 5 and the fact that the squaring \( \tilde{S}_{\lambda}^0 \) is surjective on \( F_2 \otimes_A P_k \), we have

**Theorem 6.** Let \( \ell_1, \ell_2, \ldots, \ell_{k-1} \) be integers such that \( \ell_1 > \ell_2 > \cdots > \ell_{k-1} \geq 0 \). Set \( n_r = f(\ell_1 - \ell_{r-1}, \ldots, \ell_{r-3} - \ell_{r-1}, \ell_{r-2} - \ell_{r-1} - 1, \ell_{r-2} - \ell_{r-1} - 1) \) with \( r = 3, 4, \ldots, k \). If \( \ell_{r-2} - \ell_{r-1} - 1 > i \) for \( 3 \leq i \leq k \) and \( k \geq 5 \), then

\[
\dim (F_2 \otimes_A P_k)_{2n_{r+k}} = \prod_{1 \leq i \leq k} (2^i - 1) + \sum_{5 \leq r \leq k} \left( \prod_{r+1 \leq i \leq k} (2^i - 1) \right) \dim \ker (\tilde{S}_{\lambda}^0)^r_{nk}.
\]

Here, by convention, \( \prod_{1 \leq i \leq k} (2^i - 1) = 1 \) for \( r = k \).

In order to conclude that Kameko's conjecture is false in degree \( 2n_k + k \) for any \( k \geq 5 \), it suffices to show that \( \ker (\tilde{S}_{\lambda}^0)^{f_{n_k}}_{nk} \) is nonzero. Moreover, from Theorem 5 and the inductive scheme, it is sufficient to show that the conjecture fails for \( k = 5 \).

Consider the element \( x = x_1^{\ell_1-1}x_2^{\ell_2-2}x_3^{\ell_3-1} \) in degree \( 2n_5 + 5 = f(e_1, e_2, e_3) \), where \( e_i = d_i - d_{i+1} + 1 \) for \( i = 1, 2, 3 \). The element is called a spike, i.e., a monomial whose exponents are all of the form \( 2^j - 1 \) for some \( e \).

It is well known that the class \([x]\) in \((F_2 \otimes_A P_5)_{2n_5+5}\) of a spike \( x \) is nonzero. Indeed, one has

\[
S_\alpha^{e_j} (2^j - 1 - a) = \left( \frac{2^j - 1 - a}{a} \right) x_j^{2^j - 1} = 0,
\]

as \((2^j - 1 - a) = 0\) for arbitrary \( j \) and any \( a > 0 \). Hence, a spike cannot be hit by any Steenrod operation of positive degree.

On the other hand, since the exponents of \( x_4 \) and \( x_5 \) in \( x \) are zero, \((\tilde{S}_{\lambda}^0)_{n_k}^{f_{n_k}}([x]) = 0\). Thus, we have \( \ker (\tilde{S}_{\lambda}^0)^{f_{n_k}}_{n_k} \neq 0 \). This completes the proof of Theorem 2.

However, we have \( \dim (F_2 \otimes_A P_k)_{2n_k+k} = \dim (F_2 \otimes_A P_k)_{n_k} + \dim \ker (\tilde{S}_{\lambda}^0)^{f_{n_k}}_{n_k} \). Then, by Theorem 6,

\[
\dim (F_2 \otimes_A P_k)_{n_k} = \prod_{1 \leq i \leq k} (2^i - 1) + \sum_{5 \leq r \leq k} \left( \prod_{r+1 \leq i \leq k} (2^i - 1) \right) \dim \ker (\tilde{S}_{\lambda}^0)^{f_{n_k}}_{n_k}.
\]

Hence, for \( k > 5 \), Kameko's conjecture is not true in degree \( n_k \).

The following gives our prediction on the dimension of \((F_2 \otimes_A P_k)_{2n_k+k}\) and \((F_2 \otimes_A P_k)_{n_k}\) in the appropriate case:

**Conjecture 7.** Under the hypotheses of Theorem 6,

\[
\dim \ker (\tilde{S}_{\lambda}^0)^{f_{n_k}}_{n_k} = \prod_{3 \leq i \leq k} (2^i - 1).
\]

The proofs of the results of this Note will be published in detail elsewhere.
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