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On the prescribed Q-curvature problem on S^n

Sur le problème de la Q -courbure prescrite sur Sⁿ

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ABSTRACT

In this Note we prescribe a fourth order curvature – the *Q*-curvature on the standard *n*-sphere, $n \ge 5$. Under the "flatness condition" of order β , $n - 4 \le \beta < n$ near each critical point of the prescribed *Q*-curvature function, we prove new existence result through an Euler–Hopf type formula. Our argument gives a lower bound on the number of conformal metrics having the same *Q*-curvature.

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RÉSUMÉ

Dans cette Note nous prescrivons une courbure du quatrième order-la Q-courbure sur la sphère standard de dimension $n \ge 5$. Sous une «condition de platitude» d'ordre $\beta \in [n-4, n]$ au voisinage de chaque point critique de la fonction Q-courbure prescrite, nous prouvons un nouveau résultat d'existence grâce à une formule de type Euler–Hopf. Notre argument donne une minoration du nombre des métriques ayant la même Q-courbure. © 2010 Published by Elsevier Masson SAS on behalf of Académie des sciences.

1. Introduction and the main result

Let (M^n, g_0) be a smooth compact Riemannian *n*-manifold, $n \ge 5$. We consider the Paneitz operator defined by

$$P_{g_0}^n u = \Delta_{g_0}^2 u - \operatorname{div}_{g_0}(a_n S_{g_0} g_0 + b_n Ric_{g_0}) du + \frac{n-4}{2} Q_{g_0}^n u,$$

where S_{g_0} denotes the scalar curvature of (M^n, g_0) , Ric_{g_0} denotes the Ricci curvature of (M^n, g_0) and

$$a_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)}, \qquad b_n = -\frac{4}{n-2},$$

$$Q_{g_0}^n = -\frac{1}{2(n-1)}\Delta_{g_0}S_{g_0} + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2}S_{g_0}^2 - \frac{2}{(n-2)^2}|Ric_{g_0}|^2.$$

Such a $Q_{g_0}^n$ is a fourth order invariant called Q-curvature.

The operator $P_{g_0}^n$, $n \ge 5$, is conformally invariant; if $g = u^{\frac{4}{n-4}}g_0$, u > 0 is a conformal metric to g_0 , then for all $\psi \in C^{\infty}(M)$ we have

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$$P_{g_0}^n(u\psi) = u^{\frac{n+4}{n-4}} P_g^n(\psi).$$

In particular, taking $\psi \equiv 1$, we then have

$$P_{g_0}^n(u) = \frac{n-4}{2} Q_g^n u^{\frac{n+4}{n-4}}.$$
(1)

In this work, we study the problem of prescribing Q-curvature on the standard sphere $(S^n, g_0), n \ge 5$. According to Eq. (1), the problem can be formulated as follows. Given $K \in C^1(S^n)$, we look for solutions of

$$\begin{cases} P_{g_0}^n u = \frac{n-4}{2} K u^{\frac{n+4}{n-4}}, \\ u > 0 \quad \text{on } S^n. \end{cases}$$
(2)

On the unit sphere (S^n, g_0) , $n \ge 5$, the operator $P_{g_0}^n$ is coercive on the Sobolev space $H_2^2(S^n)$ and has the expression

$$P_{g_0}^n u = \Delta_{g_0}^2 u - c_n \Delta_{g_0} u + d_n u$$

where $c_n = \frac{1}{2}(n^2 - 2n - 4)$ and $d_n = \frac{n-4}{16}n(n^2 - 4)$. Eq. (2) has a variational structure. A natural space to look in for solutions is $H_2^2(S^n)$. Due to the non-compactness of the injection of $H_2^2(S^n)$ into $L^{\frac{2n}{n-4}}(S^n)$, the Euler functional associated to (2) does not satisfy the Palais–Smale condition, which leads to the failure of the standard critical point theory.

It is easy to see that a necessary condition on K for solving Eq. (2) is that K has to be positive some where. In addition, there are topological obstructions of Kazdan-Warner type to solve (2) (see [10]). Hence, it is not expectable to solve (2) for all functions K and it is natural to ask: Under which conditions on K, problem (2) should have a solution?

Many interesting results on the Paneitz operator and related topics have been obtained by several authors; please see [1,5-8,11,12,16,17] and the references therein.

In [13], V. Felli considered the cases, when $n \ge 5$ and an Euler–Hopf type criterion for K was provided to find solutions for (2). In his paper, (please see Theorem 2.13 of [13]) the author assumed the following:

 $(f)_{\beta}$ Let $K: S^n \to \mathbb{R}$, $n \ge 5$ be a C^1 positive function such that for any critical point y of K, there exists some real number $\beta = \beta(y)$ such that in some geodesic normal coordinate system centered at y, we have

$$K(x) = K(0) + \sum_{k=1}^{n} b_k |(x)_k|^{\beta} + R(x)$$

where $b_k = b_k(y) \neq 0$, $\forall k = 1, ..., n$, $\sum_{k=1}^n b_k \neq 0$ and $\sum_{s=0}^{[\beta]} |\nabla^s R(x)| |x|^{-\beta+s} = o(1)$ as x tends to zero.

Here ∇^s denotes all possible derivatives of order *s* and [β] is the integer part of β . More precisely, she proved the following result:

Theorem 1.1. (See [13].) Assume that K satisfies $(f)_{\beta}$. If

- (i) $n 4 < \beta < n$ and $\beta \ge 2$,
- (ii) $\sum_{y \in \mathcal{K}^+} (-1)^{n \tilde{i}(y)} \neq 1$,

then (P) has at least one solution. Here, $\mathcal{K}^+ = \{y \in S^n, \nabla_{g_0} K(y) = 0 \text{ such that } -\sum_{k=1}^n b_k > 0\}$ and $\tilde{i}(y) = \sharp\{b_k, k = 1, \dots, n, N\}$ such that $b_k < 0$ }.

Let us observe that formula (ii) in the above theorem, appeared first in [14] concerning scalar curvature problem with $n-2 < \beta < n$ and in [15] when the case that $\beta = n-2$ was handled under some further condition on *K*.

In the above theorem, the author adapted the arguments of [14]. His proof involved a refined analysis for blowing-up sub-critical approximations and the use of the topological degree tools.

In this work, we restrict our attention to problem (2) under condition $(f)_{\beta}$ with $n-4 \leq \beta < n$. This leads to an interesting new phenomenon, with new result. In contrast with the case $n - 4 < \beta < n$, where only single blow-up occurs, the case $n-4 \leq \beta < n$, can present multiple blow-up points. Namely, in the case $n-4 < \beta < n$, the interaction between different bubbles dominates the self interaction of the bubbles. While, in the case $n - 4 \le \beta < n$, we can have a balance phenomenon; that is the interaction and the self interaction are of the same size.

To state our main result, we need to introduce more notations and assumptions. Let

 $\mathcal{K}_{n-4} = \{ y \in S^n, \ \nabla_{g_0} K(y) = 0 \text{ such that } \beta = \beta(y) = n-4 \}.$

For each *p*-tuple, $p \ge 1$ of distinct points $(y_{l_1}, \ldots, y_{l_p})$ such that $y_{l_i} \in \mathcal{K}^+ \cap \mathcal{K}_{n-4}$, $i = 1, \ldots, p$, we define a $p \times p$ symmetric matrix $M = M(y_{l_1}, \ldots, y_{l_p})$ by

$$m_{ii} = \frac{n-4}{n} \tilde{c}_1 \frac{-\sum_{k=1}^n b_k(y_{l_i})}{K(y_{l_i})^{\frac{n}{4}}}, \qquad m_{ij} = \tilde{c}_2 2^{\frac{n-4}{2}} \frac{-G(y_{l_i}, y_{l_j})}{[K(y_{l_i}).K(y_{l_i})]^{\frac{n-4}{8}}},$$
(3)

where $\tilde{c}_1 = \int_{\mathbb{R}^n} \frac{|x_1|^{\beta}}{(1+|x|^2)^n} dx$, $\tilde{c}_2 = \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{\frac{n+4}{2}}}$ and *G* is the Green function for the operator $P_{g_0}^n$ on S^n .

Here x_1 is the first component of x in some geodesic normal coordinates system. Let $\rho(y_{l_1}, \ldots, y_{l_n})$ be the least eigenvalue of $M(y_{l_1}, \ldots, y_{l_n})$.

(A₁) Assume that $\rho(y_{l_1}, \ldots, y_{l_p}) \neq 0$ for each distinct points $y_{l_1}, \ldots, y_{l_p} \in \mathcal{K}^+ \cap \mathcal{K}_{n-4}$.

We denote by C_{n-4}^+ the following set: $C_{n-4}^+ = \{(y_{l_1}, \ldots, y_{l_p}), p \ge 1, \text{ s.t. } y_{l_i} \in \mathcal{K}_{n-4} \cap \mathcal{K}^+ \forall i = 1, \ldots, p, y_{l_i} \neq y_{l_j} \forall i \neq j \text{ and } \rho(y_{l_1}, \ldots, y_{l_p}) > 0\}.$

Our main result is the following:

Theorem 1.2. Assume that K satisfies $(f)_{\beta}$ and (A_1) , if

(i) $n - 4 \leq \beta < n$ and $\beta \geq 2$,

(ii)
$$\sum_{y \in \mathcal{K}^+ \setminus \mathcal{K}_{n-4}} (-1)^{n-\tilde{i}(y)} + \sum_{(y_{l_1}, \dots, y_{l_p}) \in \mathcal{C}_{n-4}^+} (-1)^{p-1+\sum_{j=1}^p n-\tilde{i}(y_{l_j})} \neq 1,$$

then (2) has at least one solution.

Moreover, for generic K it holds

$$\sharp S \ge \left| 1 - \sum_{y \in \mathcal{K}^+ \setminus \mathcal{K}_{n-4}} (-1)^{n - \tilde{i}(y)} + \sum_{(y_{l_1}, \dots, y_{l_p}) \in \mathcal{C}_{n-4}^+} (-1)^{p - 1 + \sum_{j=1}^p n - \tilde{i}(y_{l_j})} \right|,$$

where *S* denotes the set of solution of (2).

Observe that our theorem does not only give existence results, but also, under generic conditions, gives a lower bound on the number of solutions of (2). Such a result is reminiscent to celebrate Morse theorem, which states that, the number of critical points of a Morse function defined on a compact manifold, is lower bounded in terms of the topology of the underlying manifold. In the following, we give a brief description on the main ingredients behind the proof of Theorem 1.2.

Our approach is completely different from the one used in [13]. We use a careful analysis of the lack of compactness of the Euler Lagrange functional *J* associated to (2). Namely we study the non-compact orbits of the gradient flow of *J* the so-called critical points at infinity following the terminology of A. Bahri [2]. These critical points at infinity can be treated as usual critical point once a Morse lemma at infinity is performed from which we can derive just as in the classical Morse theory the difference of topology induced by these non-compact orbits and compute their Morse index. Such a Morse lemma at infinity which is a cornerstone in our analysis is obtained through the construction of a suitable pseudogradient for which the Palais–Smale condition is satisfied along the decreasing flow lines as long as these flow lines do not enter the neighborhood of a finite number of critical points q_1, \ldots, q_p of *K* such that $(q_1, \ldots, q_p) \in C_{n-4}^+ \cup \mathcal{K}^+ \setminus \mathcal{K}_{n-4}$.

A similar Morse lemma has been established for the prescribed scalar curvature problem on the sphere S^3 and S^4 , under the hypothesis that the problem has no solution (see [3,4]). Since our aim is to prove multiplicity rather than only the existence we have to perform our Morse lemma without such an assumption, a situation which creates a new difficulty, namely to deal with the possibility of existence of critical point at infinity of new type consisting of a sum of bubbles plus of solution of (2). By performing a Morse lemma at infinity and by constructing a pseudogradient near an ε -neighborhood of such a potential critical point at infinity, we rule out such a possibility on S^n for any β , $n - 4 \le \beta < n$. A natural Euler– Poincaré characteristic argument allows us to derive the existence and multiplicity results. The detail of the proof of our result is given in [9].

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