Complex Analysis

Universal Taylor series for non-simply connected domains✩

Séries universelles de Taylor pour les domaines non-simplement connexes

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1. Introduction

Let $\Omega$ be a proper subdomain of the complex plane $\mathbb{C}$ and let $\zeta \in \Omega$. A function $f$ on $\Omega$ is said to belong to the collection $U(\Omega, \zeta)$, of holomorphic functions on $\Omega$ with universal Taylor series expansions about $\zeta$, if the partial sums

$$S_N(f, \zeta)(z) = \sum_{n=0}^{N} \frac{f^{(n)}(\zeta)}{n!}(z - \zeta)^n$$

of the Taylor series have the following property:

For every compact set $K \subset \mathbb{C}\setminus\Omega$ with connected complement and every function $g$ which is continuous on $K$ and holomorphic on $K^o$, there is a subsequence $(S_{N_k}(f, \zeta))$ that converges to $g$ uniformly on $K$.

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Nestoridis [17,18] has shown that $U(\Omega, \zeta) \neq \emptyset$ for any simply connected domain $\Omega$ and any $\zeta \in \Omega$. (The corresponding result, where $K$ is required to be disjoint from $\overline{\Omega}$, had previously been established by Luh [12] and Chui and Parnes [4].) In fact, Nestoridis showed that possession of such universal Taylor series expansions is a generic property of holomorphic functions on simply connected domains $\Omega$, in the sense that $U(\Omega, \zeta)$ is a dense $G_δ$ subset of the space of all holomorphic functions on $\Omega$ endowed with the topology of local uniform convergence (see also Melas and Nestoridis [14] and the survey of Kahane [11]).

The situation when $\Omega$ is non-simply connected is much less well understood, despite much recent research: see, for example [2,3,5–7,9,13,15,19,22–25]. Melas [13] (see also Costakis [5]) has shown that $U(\Omega, \zeta) \neq \emptyset$ for any $\zeta \in \Omega$ whenever $\mathbb{C} \setminus \Omega$ is compact and connected, and has asked if $U(\Omega, \zeta)$ can be empty when $\mathbb{C} \setminus \Omega$ is compact but disconnected. On the other hand, Müller, Vlachou and Yavrian [15] have shown, for non-simply connected domains $\Omega$, that thinness of the set $\mathbb{C} \setminus \Omega$ at infinity is necessary for $U(\Omega, \zeta)$ to be non-empty, and have conjectured that this condition is also sufficient. There is clearly a large gap between the results of [13] and [15]. Also there has been no known example of a domain $\Omega$ and points $\zeta_1, \zeta_2 \in \Omega$ such that $U(\Omega, \zeta_1) \neq \emptyset$ and $U(\Omega, \zeta_2) = \emptyset$.

The purpose of this Note is to establish the following result. We denote by $D(a, r)$ the open disc of centre $a$ and radius $r$, and write $\mathbb{D} = D(0, 1)$. By a non-degenerate continuum we mean a connected compact set containing more than one element.

**Theorem 1.** Let $\Omega$ be a domain of the form $\mathbb{C} \setminus (L \cup \{0\})$, where $L$ is a non-degenerate continuum in $\mathbb{C} \setminus \mathbb{D}$. Then $U(\Omega, 0) = \emptyset$.

The conjecture of Müller, Vlachou and Yavrian is thus disproved. Also, if we take $L$ to be $\overline{\mathbb{D}}(-5/3, 1/3)$, then $U(\Omega, 0) = \emptyset$ by Theorem 1 and yet a result of the second author [22] tells us that $U(\Omega, -1/2) \neq \emptyset$ (see also Costakis and Vlachou [7]). Thus we now have an example of a domain $\Omega$ where the existence of functions with universal Taylor series depends on the chosen centre for expansion. The result of Melas, that $U(\Omega, 0) \neq \emptyset$ if $\mathbb{C} \setminus \Omega$ is compact and connected, is now seen to be sharp in the sense that, by Theorem 1, it can fail with the removal of one additional point from the domain. Theorem 1 fails if $L$ is allowed to be a singleton [13].

**2. Proof**

Let $\Omega$ be as in the statement of Theorem 1, and suppose, for the sake of contradiction, that there exists a function $f$ in $U(\Omega, 0)$. We can write $f = g + h$, where $g$ is the singular part of the Laurent expansion of $f$ associated with the singularity at 1, and $h$ is holomorphic on $\mathbb{C} \setminus L$. We denote the Taylor coefficients of $g$ and $h$ about 0 by $(a_n)$ and $(b_n)$, respectively.

Since $(S_k(f, 0)(1))$ is dense in $\mathbb{C}$ and $(S_k(h, 0)(1))$ converges, we see that $g$ is non-zero.

Let $\rho = \text{inf}\{|z| \in \mathbb{D} : 0 < \delta < \varepsilon < \rho - 1\}$. The Taylor series for $g$ and $h$ about 0 converge absolutely in $\mathbb{D}$ and $D(0, \rho)$, respectively, so we can define the finite quantities

$$\alpha_\delta = \sum_{n=0}^{\infty} \frac{|a_n|}{(1 + \delta)^n} \quad \text{and} \quad \beta_\delta = \sum_{n=0}^{\infty} |b_n| \left(\frac{\rho}{1 + \delta}\right)^n.$$

Since $f \in U(\Omega, 0)$, we can choose a strictly increasing sequence $(N_k)$ of natural numbers such that

$$S_{N_k}(g, 0)(z) + S_{N_k}(h, 0)(z) \to 0 \quad \text{as} \ k \to \infty, \quad \text{uniformly on} \ L.$$

(1)

On $D(0, \rho(1 + \varepsilon))$ we have

$$|S_{N_k}(h, 0)(z)| \leq \sum_{n=0}^{N_k} |b_n| \rho^n (1 + \varepsilon)^n \leq \left((1 + \varepsilon)(1 + \delta)^n\right)^{N_k} \beta_\delta,$$

so by (1) we can choose $k_0$ such that

$$|S_{N_k}(g, 0)(z)| \leq \left((1 + \varepsilon)(1 + \delta)^n\right)^{N_k} (\beta_\delta + 1) \quad (z \in L \cap \overline{D}(0, \rho(1 + \varepsilon)); \ k \geq k_0).$$

We also have

$$|S_{N_k}(g, 0)(z)| \leq \sum_{n=0}^{N_k} |a_n|(1 + \varepsilon)^n \leq \left((1 + \varepsilon)(1 + \delta)^n\right)^{N_k} \alpha_\delta \quad (z \in D(0, 1 + \varepsilon)).$$

so

$$|S_{N_k}(g, 0)(z)| \leq \left((1 + \varepsilon)(1 + \delta)^n\right)^{N_k} \gamma_\delta \quad (z \in A_\varepsilon; \ k \geq k_0),$$

where $\gamma_\delta = \max(\alpha_\delta, \beta_\delta + 1)$ and $A_\varepsilon = \overline{D}(0, 1 + \varepsilon) \cup \{z \in D(0, \rho(1 + \varepsilon))\}$.

Let $G_\varepsilon$ denote the Green function for the domain $D_\varepsilon = (\mathbb{C} \cup \{\infty\}) \setminus A_\varepsilon$ with pole at infinity. Then

$$G_\varepsilon(z) - \log|z| \to -\log C(A_\varepsilon) \quad (|z| \to \infty),$$

where $C(A_\varepsilon)$ is the capacity of $A_\varepsilon$. This concludes the proof of Theorem 1.
where \( C(A) \) denotes the logarithmic capacity of a set \( A \) (see Section 5.8 of [1], or Section 5.2 of [21]). Thus we can choose \( r_{\delta, \varepsilon} > \max \{|z|: \ z \in L\} \) such that
\[
G_\varepsilon(z) \leq \log |z| - \log C(A_e) + \delta \quad (|z| \geq r_{\delta, \varepsilon}).
\] (3)

Bernstein’s lemma (Theorem 5.5.7 in [21]) tells us that any polynomial \( q \) of degree \( n \geq 1 \) satisfies
\[
\left( \frac{|q(z)|}{\max_{A_e} |q|} \right)^{1/n} \leq e^{G_\varepsilon(z)} \quad (z \in D_e \setminus \{\infty\}).
\]

Applying this inequality to the polynomial \( S_{N_k}(g, 0) \), and using (2) and then (3), we obtain
\[
|S_{N_k}(g, 0)(z)| \leq \left\{ (1 + \varepsilon)(1 + \delta) \right\}^{N_k} \gamma \frac{e^{N_k G_\varepsilon(z)}}{C(A_e)} \quad (|z| \geq r_{\delta, \varepsilon}; \ k \geq k_0).
\]

We next adapt an argument from pp. 498, 499 of Gehlen [8]. Let \( v \in (0, 1) \). Since
\[
|a_n|^{1/n} = \left| \frac{1}{2\pi i} \int_{|z|=\delta, \varepsilon} \frac{S_{N_k}(g, 0)(z)}{z^{n+1}} \ dz \right|^{1/n}
\]
\[
\leq \left\{ (1 + \varepsilon)(1 + \delta) \right\}^{N_k/n} \gamma^{1/n} \frac{r_{\delta, \varepsilon}^{1/n} e^{N_k/n - 1}}{C(A_e)} \quad (n \leq N_k; \ k \geq k_0),
\]
we obtain
\[
\limsup_{k \to \infty} \max_{N_k \leq n \leq N_k} |a_n|^{1/n} \leq \frac{((1 + \varepsilon)(1 + \delta) \varepsilon)^{1/n}}{C(A_e)} = \lambda, \quad \text{say}.
\] (4)

Since \( L \) is a non-degenerate continuum that intersects \( \{|z| = \rho\} \), we have
\[ C(L \cap \overline{D}(0, \rho(1 + \varepsilon))) > 0 \]
and so
\[ C(A_e) > C(\overline{D}(0, 1 + \varepsilon)) = 1 + \varepsilon. \]

We can thus choose \( \delta \) sufficiently small that \( (1 + \varepsilon)(1 + \delta) \varepsilon < C(A_e) \), and then choose \( v \) sufficiently close to 1 to ensure that \( \lambda < 1 \).

Finally, we will apply an observation of Müller (see Remark 2 in [16]). Since the function \( g \) has its only singularity at 1 and vanishes at \( \infty \), Wigert’s theorem (Theorem 11.2.2 in Hille [10]) tells us that there is an entire function \( F \) of exponential type 0 such that \( F(n) = a_n \) for all \( n \geq 0 \). However, Theorem V of Pólya [20] says that, for any \( \mu > 0 \), however small, such a function \( F \) has the property that the sequence \( \{n \in \mathbb{N}: F(n) > e^{\mu n}\} \) is of density 1. This contradicts (4) with \( \lambda < 1 \). Thus our original assumption, that there exists \( f \) in \( U(\Omega, 0) \), must be false, and the proof of the theorem is complete. \( \square \)

Remarks.

(1) The assumption that \( L \) is a continuum can be relaxed. It is enough to suppose that \( L \) is a compact subset of \( \mathbb{C} \setminus \overline{D} \) such that \( C(D(0, \rho^2) \cap L) > 0 \) where \( \rho = \inf \{|z|: \ z \in L\} \).

(2) The proof actually shows that there is no holomorphic function \( f \) on \( \Omega \) such that \( (S_N(f, 0)) \) is divergent at \( z = 1 \) and has a subsequence that is uniformly bounded on \( L \).

References


