Group Theory

On representation zeta functions of groups and a conjecture of Larsen–Lubotzky

Sur les fonctions zêta de représentations de groupes et sur une conjecture de Larsen–Lubotzky

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We study zeta functions enumerating finite-dimensional irreducible complex linear representations of compact \(p\)-adic analytic and of arithmetic groups. Using methods from \(p\)-adic integration, we show that the zeta functions associated to certain \(p\)-adic analytic pro-\(p\) groups satisfy functional equations. We prove a conjecture of Larsen and Lubotzky regarding the abscissa of convergence of arithmetic groups of type \(A_2\) defined over number fields, assuming a conjecture of Serre on lattices in semisimple groups of rank greater than 1.

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RÉSUMÉ

On étudie les fonctions zêta dénombrant les représentations linéaires complexes irréductibles de dimension finie de groupes compacts \(p\)-adiques analytiques et de groupes arithmétiques. En utilisant une méthode d'intégration \(p\)-adique, on démontre que celles de ces fonctions qui sont associées à certains pro-\(p\)-groupes \(p\)-adiques analytiques satisfont à des équations fonctionnelles. En admettant une conjecture de Serre sur les réseaux dans les groupes semi-simples de rang supérieur 1, on démontre une conjecture de Larsen et Lubotzky pour les groupes algébriques de type \(A_2\) définis sur des corps de nombres.

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Soit \(G\) un groupe et, pour tout \(n \in \mathbb{N}\), soit \(r_n(G)\) le nombre de classes d’équivalence de représentations linéaires complexes irréductibles de dimension \(n\) de \(G\); si \(G\) est un groupe topologique (resp. un groupe algébrique), on suppose que les représentations sont continues (resp. rationnelles). Si \(G\) est rigide, c’est-à-dire si \(r_n(G)\) est fini pour tout \(n \in \mathbb{N}\), il est naturel d’étudier la suite \(r_n(G)\) et son comportement asymptotique quand \(n\) tend vers l’infini. Si, de plus, \(G\) est de croissance de représentation polynomiale, c’est-à-dire si \(r_n(G)\) est borné par un polynôme en \(n\), un outil puissant pour l’étude de la croissance de représentations est la fonction zêta de représentations associée.
where \( s \) is a complex variable. It is well known that the \textit{abscissa of convergence} \( \alpha(G) \) of the series \( \zeta_G(s) := \sum_{n=1}^{\infty} r_n(G) n^{-s} \),

where \( s \) is a complex variable. It is well known that the \textit{abscissa of convergence} \( \alpha(G) \) of the series \( \zeta_G(s) \), i.e. the infimum of all \( \alpha \in \mathbb{R} \) such that \( \zeta_G(s) \) converges on the complex right half-plane \( \{ s \in \mathbb{C} \mid \text{Re}(s) > \alpha \} \), gives the precise degree of polynomial growth: \( R_N(G) = O(1 + N^{\alpha+\epsilon}) \) for every \( \epsilon \in \mathbb{R}_{>0} \).
In [2] we introduce new methods from the theory of $p$-adic integration to study representation zeta functions associated to compact $p$-adic analytic groups and arithmetic groups. In [3] we compute explicit formulae for the representation zeta functions of the groups $SL_3(o)$, where $o$ is a compact discrete valuation ring of characteristic 0, in the case that the residue field characteristic is large compared to the ramification index of $o$.

A finitely generated profinite group $G$ is rigid if and only if it is FAb, i.e. if every open subgroup of $G$ has finite abelianisation. In [5], Jaikin-Zapirain proved rationality results for the representation zeta functions of FAb compact $p$-adic analytic groups using tools from model theory. In particular, the representation zeta function of a FAb $p$-adic analytic pro-$p$ group is a rational function in $p^{-s}$, for $p > 2$. Key examples of FAb compact $p$-adic analytic groups are the special linear groups $SL_n(o)$ and their principal congruence subgroups $SL_n^m(o)$, where $o$ is a compact discrete valuation ring of characteristic 0 and residue field characteristic $p$. For fixed $n$, and varying $m$ and $o$, the latter also yield important examples of families of pro-$p$ groups which arise from a global Lie lattice, in this case $g_m(E)$.

To be more precise, let $O$ be the ring of integers of a number field $k$, and let $A$ be an $O$-Lie lattice such that $k \otimes O A$ is a perfect $k$-Lie algebra of dimension $d$. Let $o = O_v$ be the ring of integers of the completion $k_v$ of $k$ at a non-archimedean place $v$, lying above a rational prime $p$. Given a finite extension $D$ of $o$, we write $\mathcal{P}$ for the maximal ideal of $D$, $e(D)o$ for the ramification index and $f(D)o$ for the residue class field extension degree. Let $g(D) := O \otimes O A$. For all sufficiently large $m$, the Lie lattice $g^{m}(D) := \mathcal{P}^m g(D)$ corresponds, by $p$-adic Lie theory, to a FAb, potent, saturable pro-$p$ group $G^m(D) := \exp(g^{m}(D))$. We call such $m$ permissible for the Lie lattice $g(D)$. For example, for unramified extensions $D$ of $Z_p$ and $p$ odd, every $m \in \mathbb{N}$ is permissible. In [2], we prove

**Theorem A.** In the above setup, there exists a finite set $S$ of places of $k$, a natural number $r$ and a rational function $W(X_1, \ldots, X_r, Y) \in \mathbb{Q}(X_1, \ldots, X_r, Y)$ such that, for every non-archimedean place $v$ of $k$ with $v \notin S$, the following is true:

There exist algebraic integers $\lambda_1 = \lambda_1(v), \ldots, \lambda_d = \lambda_d(v)$ such that for all finite extensions $D$ of $o = O_v$ and for all $m \in \mathbb{N}$ which are permissible for $g(D)$ one has

\[
\zeta_{G^m(D)}(s) = q_v^{f(m)} W(\lambda_1, \ldots, \lambda_d, q_v^{-fs}),
\]

where $q_v$ denotes the residue field cardinality of $o$, $f = f(D)o$ and $d = \text{rank}_D(g(D)) = \dim_k(k \otimes O A)$. Furthermore, the functional equation

\[
\zeta_{G^m(D)}(s) \mid_{q_v \rightarrow q_v^{-1}} = q_v^{d(1-2m)} \zeta_{G^m(D)}(s)
\]

holds.

Our proof of Theorem A implies in particular that the real parts of the poles of the zeta functions $\zeta_{G^m(D)}(s)$ are rational numbers. More precisely, we prove the following:

**Theorem B.** In the above setup, there exists a finite set $P \subset \mathbb{Q}$ such that for all non-archimedean places $v$ of $k$, all finite extensions $D$ of $o = O_v$, all permissible $m$ for $g(D)$ one has

\[
\{ \text{Re}(s) \mid s \text{ a pole of } \zeta_{G^m(D)}(s) \} \subseteq P.
\]

Furthermore, if $v \notin S$, where $S$ is a finite set of places arising from Theorem A, and if $O_v \subseteq D_1 \subseteq D_2$, then for every $m \in \mathbb{N}$ which is permissible for $g(D_1)$ and $g(D_2)$,

\[
\alpha(G^m(D_1)) \leq \alpha(G^m(D_2)).
\]

Notice that, if the groups $G^m(D)$ are principal congruence subgroups of a FAb compact $p$-adic analytic group $G(D)$ consisting of the $\mathcal{O}$-points of an algebraic group $G$, such as $G = SL_n$, then (3) implies the monotonicity of the abscissae of convergence $\alpha(G(D))$ under ring extensions. This follows from the fact that these abscissae are commensurability invariants.

The set $P$ of candidate poles is obtained by means of a resolution of singularities which leads to the generic formula (1). Theorems A and B are illustrated by the explicit formulae given in Theorem E below. The proofs of Theorems A and B are based on the Kirillov orbit method for the groups in question [4,5], and methods from $p$-adic integration [8,9].

The arithmetic groups we are interested in are arithmetic lattices in semisimple algebraic groups defined over number fields. More precisely, let $G$ be a connected, simply connected semisimple algebraic group, defined over a number field $k$, together with a fixed $k$-embedding into some $GL_n$. Let $O_5$ denote the ring of $S$-integers in $k$, for a finite set $S$ of places of $k$ including all the archimedean ones. We consider groups $\Gamma$ which are commensurable to $G(O_5) = G(k) \cap GL_n(O)$.

In [7], Lubotzky and Martin showed that such a group $\Gamma$ has PRG if and only if $\Gamma$ has the Congruence Subgroup Property (CSP). Suppose that $\Gamma$ has these properties. Then, according to a result of Larsen and Lubotzky [6, Proposition 1.3], the representation zeta function of $\Gamma$ admits an Euler product decomposition. Indeed, if $\Gamma = G(O_5)$ and if the congruence kernel of $\Gamma$ is trivial, this decomposition is particularly easy to state: it takes the form
\[ \zeta_T(s) = \zeta_{\text{G}(C)}(s)^{[\mathbb{C}:\mathbb{Q}]} \prod_{v \in S} \zeta_{\text{G}(O_v)}(s). \]  

(4)

Here each archimedean factor \( \zeta_{\text{G}(C)}(s) \) enumerates rational representations of the group \( \text{G}(\mathbb{C}) \); their contribution to the Euler product reflects Margulis super-rigidity. The groups \( \text{G}(O_v) \) are \( p \)-adic analytic groups whose principal congruence subgroups fit into the framework of Theorems A and B; the product of the zeta functions of these local groups captures the finite image representations of \( T \).

Several of the key results of [6] concern the abscissae of convergence of the ‘local’ representation zeta functions occurring as Euler factors on the right-hand side of (4). With regards to abscissae of convergence of the ‘global’ representation zeta functions Avni proved that, for an arithmetic group \( T \) with the CSP, the abscissa of convergence of \( \zeta_T(s) \) is always a rational number; see [1]. In [6, Conjecture 1.5], Larsen and Lubotzky conjectured that, for any two irreducible lattices \( T_1 \) and \( T_2 \) in a higher-rank semisimple group \( H \), one has \( \alpha(T_1) = \alpha(T_2) \), i.e. that the abscissa of convergence only depends on the ambient group. This can be regarded as a refinement of Serre’s conjecture on the Congruence Subgroup Property. In [6, Theorem 10.1], Larsen and Lubotzky prove their conjecture in the case that \( H \) is a product of simple groups of type \( A_1 \), assuming Serre’s conjecture. In [2], we prove

**Theorem C.** Let \( T \) be an arithmetic lattice of a connected, simply connected simple algebraic group of type \( A_2 \) defined over a number field. If \( T \) has the CSP, then \( \alpha(T) = 1 \).

**Corollary D.** Assuming Serre’s conjecture, Larsen and Lubotzky’s conjecture holds for groups of the form \( H = \prod_{i=1}^r G_i(K_i) \), where each \( K_i \) is a local field of characteristic 0 and each \( G_i \) is an absolutely almost simple \( K_i \)-group of type \( A_2 \) such that \( \sum_{i=1}^r \text{rk}_{K_i}(G_i) \geq 2 \) and none of the \( G_i(K_i) \) is compact.

Key to our proof of Theorem C is the following local result, which we formulate in accordance with the notation introduced before Theorem A:

**Theorem E.** Let \( \mathcal{O} \) be a compact discrete valuation ring of characteristic 0, with residue field of cardinality \( q \). Let \( g(\mathcal{O}) \) be one of the following two \( \mathcal{O} \)-Lie lattices of type \( A_2 \):

(a) \( sl_3(\mathcal{O}) = \{ x \in g(sl_3(\mathcal{O}) | \text{Tr}(x) = 0 \};
(b) \( su_3(\mathcal{O}, \mathcal{O}) = \{ x \in g(sl_3(\mathcal{O}) | x^3 = -x \}, \) where \( \mathcal{O} \mathcal{O} \) is an unramified quadratic extension with nontrivial automorphism \( \sigma \).

For \( m \in \mathbb{N} \), let \( G^m(\mathcal{O}) \) be the \( m \)-th principal congruence subgroup of the corresponding group \( SL_3(\mathcal{O}) \) or \( SU_3(\mathcal{O}, \mathcal{O}) \). Assume that the residue field characteristic of \( \mathcal{O} \) is not equal to 3. Then, for all \( m \in \mathbb{N} \) which are permissible for \( g(\mathcal{O}) \), one has

\[ \zeta_{G^m(\mathcal{O})}(s) = q^{8m} \frac{1 + u(q)q^{-32s} + u(q^{-1})q^{-2-3s} + q^{-5-5s}}{(1 - q^{1-2s})(1 - q^{2-3s})}, \]

where

\[ u(X) = \begin{cases} X^3 + X^2 - X - 1 & , \text{if } g(\mathcal{O}) = sl_3(\mathcal{O}), \\ -X^3 + X^2 - X + 1 & , \text{if } g(\mathcal{O}) = su_3(\mathcal{O}, \mathcal{O}). \end{cases} \]

The close resemblance between the representation zeta functions of the special linear and the special unitary groups is noteworthy and reminiscent of the Ennola duality for the characters of the corresponding finite groups of Lie type. We also give an explicit formula for \( \zeta_{SL_3(\mathcal{O})}(s) \) in the exceptional case where \( \mathcal{O} \) is unramified and has residue field characteristic 3. Note that Theorem E implies that the abscissae of convergence \( \alpha(SL_3(\mathcal{O})) \) and \( \alpha(SU_3(\mathcal{O}, \mathcal{O})) \) are each equal to 2/3, as the abscissa of convergence is a commensurability invariant. Theorem E is proved using the techniques from \( p \)-adic integration which allow us to establish Theorems A and B.

The explicit formula for \( \zeta_{SL_3(\mathcal{O})}(s) \), which we present in [3], is deduced by means of the Kirillov orbit method, a description of the similarity classes in finite quotients of \( g(sl_3(\mathcal{O})) \), and Clifford theory.

**Theorem F.** There exist finitely many polynomials \( f_{i,j}, g_{i,j} \in \mathbb{Q}[x] \), indexed by \((i,j) \in \{1, -1\} \times I \), such that for every compact discrete valuation ring \( \mathcal{O} \) of characteristic 0, with residue field characteristic \( p > 3e(\mathcal{O}|\mathbb{Z}_p) \), one has

\[ \zeta_{SL_3(\mathcal{O})}(s) = \sum_{i \leq 1} f_{i,j}(q)(g_{i,j}(q))^{-\ell} \frac{(1 - q^{1-2s})(1 - q^{2-3s})}{(1 - q^{1-2s})(1 - q^{2-3s})}, \]

where \( q \) denotes the size of the residue field of \( \mathcal{O} \) and \( q \equiv \ell \pmod{3} \).

An explicit set of such polynomials \( f_{i,j}, g_{i,j} \) is computed in [3]. This result should be seen against the background of [5, Theorem 1.1], which establishes the rationality of representation zeta functions of \( F_{\text{Ab}} \) compact \( p \)-adic analytic groups. For
groups of the form $\text{SL}_3(o)$, Theorem F specifies that these rational functions vary ‘uniformly’ as a function of the residue field cardinality $q$. Theorem F enables us to analyse the global representation zeta functions of the arithmetic groups $\text{SL}_3(O_S)$.

**Theorem G.** Let $O_S$ be the ring of $S$-integers of a number field $k$, where $S$ is a finite set of places of $k$ including all the archimedean ones. Then there exists $\epsilon > 0$ such that the representation zeta function of $\text{SL}_3(O_S)$ admits a meromorphic continuation to the half-plane \( \{ s \in \mathbb{C} \mid \text{Re}(s) > 1 - \epsilon \} \). The continued function is holomorphic on the line \( \{ s \in \mathbb{C} \mid \text{Re}(s) = 1 \} \) except for a double pole at $s = 1$. There is a constant $c \in \mathbb{R} > 0$ such that

\[
\sum_{n=1}^{N} r_n(\text{SL}_3(O_S)) \sim c \cdot N \log N,
\]

where $f(N) \sim g(N)$ means $\lim_{N \to \infty} f(N)/g(N) = 1$.

In [2], we also give simple geometric estimates for the abscissae of convergence of representation zeta functions of compact $p$-adic analytic groups and we compute representation zeta functions associated to norm-1 groups in non-split quaternion algebras.

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**References**


