Partial Differential Equations/Differential Geometry

# Q-curvature flow with indefinite nonlinearity ${ }^{\text {at }}$ 

## Flot de Q-courbure pour une non-linéarité indéfinie

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## A R T I C L E IN F O

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#### Abstract

In this Note, we study Q-curvature flow on $S^{4}$ with indefinite nonlinearity. Our result is that the prescribed Q-curvature problem on $S^{4}$ has a solution provided the prescribed non-negative Q-curvature $f$ has its positive part, which possesses non-degenerate critical points such that $\Delta_{S^{4}} f \neq 0$ at the saddle points and an extra condition such as a nontrivial degree counting condition. © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

\section*{R É S U M É}

Dans cette Note on étudie le flot de Q-courbure sur $S^{4}$ dans le cas d'une non-linéarité indéfinie. Le résultat montre que le problème de la $Q$-courbure imposée sur $S^{4}$ a une solution à condition que la Q -courbure non négative imposée $f$ ait une partie strictement positive et des points critiques non dégénérés tels que $\Delta_{S^{4}} f \neq 0$ aux points selles et une condition supplémentaire du type condition non triviale sur le degré.


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## 1. Introduction

Following the works of A. Chang and P. Yang [5], M. Brendle [4], A. Malchiodi and M. Struwe [9], we study a heat flow method to the prescribed Q-curvature problem on $S^{4}$. Given the Riemannian metric $g$ in the conformal class of standard metric $c$ on $S^{4}$ with $Q$-curvature $Q_{g}$. It is well known that

$$
Q_{g}=-\frac{1}{12}\left(\Delta_{g} R_{g}-R_{g}^{2}+3|R c(g)|^{2}\right):=Q
$$

where $R_{g}, R c(g), \Delta_{g}$ are the scalar curvature, Ricci curvature tensor, the Laplacian operator of the metric $g$, respectively.
Recall the Chern-Gaussian-Bonnet formula on $S^{4}$ is,

$$
\int_{S^{4}} Q_{g} \mathrm{~d} v_{g}=8 \pi^{2}
$$

By this, we know that $Q_{g}$ has to be positive somewhere. This gives a necessary condition for the prescribed Q-curvature problem on $S^{4}$. Assuming the prescribed curvature function $f$ being positive on $S^{4}$, the heat flow for the Q-curvature problem is a family of metrics of the form $g=e^{2 u(x, t)} c$ satisfying:

[^0]\[

$$
\begin{equation*}
u_{t}=\alpha f-Q, \quad x \in S^{4}, t>0 \tag{1}
\end{equation*}
$$

\]

where $u: S^{4} \times(0, T) \rightarrow R$ is the unknown, and $\alpha=\alpha(t)$ is defined by

$$
\begin{equation*}
\alpha \int_{S^{4}} f \mathrm{~d} v_{g}=8 \pi^{2} \tag{2}
\end{equation*}
$$

Here $\mathrm{d} v_{g}$ is the area element with respect to the metric $g$. It is easy to see that $\alpha_{t} \int_{S^{4}} f \mathrm{~d} v_{g}=2 \alpha \int_{S^{4}}(Q-\alpha f) f \mathrm{~d} v_{g}$. A. Malchiodi and M. Struwe [9] can show in their Theorem 1.1 that the flow exists globally, furthermore, the flow converges at time infinity provided $f$ is positive and possesses non-degenerate critical points such that $\Delta_{S^{4}} f \neq 0$ at the saddle points with the condition,

$$
\sum_{\left\{p: \nabla f(p)=0 ; \Delta_{s^{4}} f(p)<0\right\}}(-1)^{\text {ind }(f, p)} \neq 0
$$

Here $\Delta_{S^{4}}:=\Delta$ is the analyst's Laplacian on the standard 4 -sphere $\left(S^{4}, c\right)$. Recall that $\int_{S^{4}} \mathrm{~d} v_{c}=\frac{8}{3} \pi^{2}$. The purpose of this Note is to relax their assumption by allowing the function $f$ to have zeros.

Since we have

$$
Q=\frac{1}{2} e^{-4 u}\left(\Delta^{2} u-\operatorname{div}\left(\left(\frac{2}{3} R(c) c-2 R c(c)\right) d u\right)+6\right)
$$

Eq. (1) defines a nonlinear parabolic equation for $u$, and the flow exists at least locally for any initial data $\left.u\right|_{t=0}=u_{0}$ and any smooth function $f$ being positive somewhere. Clearly, we have:

$$
\partial_{t} \int_{S^{4}} \mathrm{~d} v_{g}=2 \int_{S^{4}} u_{t} \mathrm{~d} v_{g}=0
$$

We shall assume that the initial data $u_{0}$ satisfies the condition,

$$
\begin{equation*}
\int_{S^{4}} f e^{4 u} \mathrm{~d} v_{c}>0 \tag{3}
\end{equation*}
$$

We remark that since $f$ can be approximated by positive smooth functions, the set of functions satisfying (3) should be contractible. Then we can use the handle-body theorem following Malchiodi and Struwe [9]. We shall show that the property (3) is preserved along the flow even for $f$ changing signs. In some sense, this may be known to experts. It is easy to compute that

$$
\begin{equation*}
Q_{t}=-4 u_{t} Q-\frac{1}{2} P u_{t}=4 Q(Q-\alpha f)+P(\alpha f-Q) \tag{4}
\end{equation*}
$$

where $P=P_{g}=e^{-4 u} P_{c}$ and $P_{c}$ is the Paneitz operator in the metric $c$ on $S^{4}$ [5]. Using (4), we can compute the growth rate of the Calabi-type energy $\int_{S^{4}}|Q-\alpha f|^{2} \mathrm{~d} v_{g}$.

Our main result is following:
Theorem 1. Let $f$ be a positive somewhere, non-negative smooth function on $S^{4}$ with only non-degenerate critical points on the its positive part $f_{+}$with its Morse index ind $\left(f_{+}, p\right)$. Suppose that at each critical point $p$ of $f_{+}$, we have $\Delta f \neq 0$. Let $m_{i}$ be the number of critical points with $f(p)>0, \Delta_{S^{4}} f(p)<0$ and ind $(f, p)=4-i$. Suppose that there is no solutions with coefficients $k_{i} \geqslant 0$ to the system of equations

$$
m_{0}=1+k_{0}, \quad m_{i}=k_{i-1}+k_{i}, \quad 1 \leqslant i \leqslant 4, k_{4}=0
$$

Then $f$ is the $Q$-curvature of a conformal metric $g=e^{2 u} c$ on $S^{4}$.
A similar result for curvature flow to the Nirenberg problem on $S^{2}$ has been obtained in [6]. See [2,7,11] and [8] for related.

For simplifying notations, we shall use the conventions that $\mathrm{d} c=\frac{\mathrm{d} v_{c}}{\frac{8}{3} \pi^{2}}$ and $\bar{u}=\bar{u}(t)$ defined by: $\int_{S^{4}}(u-\bar{u}) \mathrm{d} v_{c}=0$.

## 2. Basic properties of the flow

In this section we may allow $f$ to change signs. Recall the following result of Beckner [3]:

$$
\begin{equation*}
\int_{S^{4}}\left(|\Delta u|^{2}+2|\nabla u|^{2}+12 u\right) \mathrm{d} c \geqslant \log \left(\int_{S^{4}} e^{4 u} \mathrm{~d} c\right)=0 \tag{5}
\end{equation*}
$$

where $|\nabla u|^{2}$ is the norm of the gradient of the function $u$ with respect to the standard metric $c$. Here we have used the fact that $\int_{S^{4}} e^{4 u} \mathrm{~d} c=1$ along the flow (1).

We show that the condition (3) is preserved along the flow (1). In fact, letting $E(u)=\int_{S^{4}}\left(u P u+4 Q_{c} u\right) \mathrm{d} c=\int_{S^{4}}\left(|\Delta u|_{c}^{2}+\right.$ $\left.2|\nabla u|_{c}^{2}+12 u\right) \mathrm{d} c$ be the Liouville energy of $u$ and letting, $E_{f}(u)=E(u)-3 \log \left(\int_{S^{4}} f e^{4 u} \mathrm{~d} c\right)$, be the energy function for the flow (1), we then compute that

$$
\begin{equation*}
\partial_{t} E_{f}(u)=-\frac{3}{2 \pi^{2}} \int_{S^{4}}|\alpha f-Q|^{2} \mathrm{~d} v_{g} \leqslant 0 \tag{6}
\end{equation*}
$$

One may see Lemma 2.1 in [9] for a proof of this formula. Hence

$$
E_{f}(u(t)) \leqslant E_{f}\left(u_{0}\right), \quad t>0
$$

After using the inequality (5) we have,

$$
\begin{equation*}
\log \left(1 / \int_{S^{4}} f e^{4 u} \mathrm{~d} c\right) \leqslant E_{f}\left(u_{0}\right) \tag{7}
\end{equation*}
$$

which implies that $\int_{S^{4}} f e^{4 u} \mathrm{~d} v_{c}>0$ and furthermore, $e^{E_{f}\left(u_{0}\right)} \int_{S^{4}} e^{4 u} \mathrm{~d} c \leqslant \int_{S^{4}} f e^{4 u} \mathrm{dc}$.
Note also that $\int_{S^{4}} f e^{4 u} \mathrm{~d} c=1 / \alpha(t)$. Hence, $\alpha(t) \leqslant \frac{1}{e^{E_{f}\left(u_{0}\right)}}$. Using the definition of $\alpha(t)$ we have: $\alpha(t) \geqslant \frac{1}{\max _{s^{4} f}}$. We then conclude that $\alpha(t)$ is uniformly bounded along the flow, i.e.,

$$
\begin{equation*}
\frac{1}{\max _{S^{4}} f} \leqslant \alpha(t) \leqslant \frac{1}{e^{E_{f}\left(u_{0}\right)}} \tag{8}
\end{equation*}
$$

We shall use this inequality to replace (26) in [9] in the study of the normalized flow, which will be defined in the next section following the work of A. Malchiodi and M. Struwe [9]. If we have a global Q-curvature flow, then using (6) we have:

$$
2 \int_{0}^{\infty} \mathrm{d} t \int_{S^{4}}|\alpha f-Q|^{2} \mathrm{~d} v_{g} \leqslant 4 \pi\left(E_{f}\left(u_{0}\right)+\log \max _{S^{4}} f\right)
$$

Hence we have a suitable sequence $t_{l} \rightarrow \infty$ with associated metrics $g_{l}=g\left(t_{l}\right)$ and $\alpha\left(t_{l}\right) \rightarrow \alpha>0$, and letting $Q_{l}=Q\left(g_{l}\right)$ be the Q-curvature of the metric $g_{l}$, such that $\int_{S^{4}}\left|Q_{l}-\alpha f\right|^{2} \rightarrow 0\left(t_{l} \rightarrow \infty\right)$. Therefore, once we have a limiting metric $g_{\infty}$ of the sequence of the metrics $g_{l}$, it follows that $Q\left(g_{\infty}\right)=\alpha f$. After a re-scaling, we see that $f$ is the Q-curvature of the metric $\beta g_{\infty}$ for some $\beta>0$, which implies our Theorem 1 .

## 3. Normalized flow and the proof of Theorem 1

In this section, we fix $f$ assumed in Theorem 1. We now introduce the normalized flow. For the given flow $g(t)=$ $e^{2 u(t)} c$ on $S^{4}$, there exists a family of conformal diffeomorphisms $\phi=\phi(t): S^{4} \rightarrow S^{4}$, which depends smoothly on the time variable $t$, such that for the metrics $h=\phi^{*} g$, we have:

$$
\int_{S^{4}} x \mathrm{~d} v_{h}=0, \quad \text { for all } t \geqslant 0
$$

Here $x=\left(x^{1}, x^{2}, x^{3}, x^{4}, x^{5}\right) \in S^{4} \subset R^{5}$ is a position vector of the standard 4 -sphere. Let $v=u \circ \phi+\frac{1}{4} \log (\operatorname{det}(d \phi))$. Then we have $h=e^{2 v} c$. Using the conformal invariance of the Liouville energy [5], we have: $E(v)=E(u)$, and furthermore, $\operatorname{Vol}\left(S^{4}, h\right)=\operatorname{Vol}\left(S^{4}, g\right)=\frac{8}{3} \pi^{2}$, for all $t \geqslant 0$.

Assume $u(t)$ satisfies (1) and (2). Then we have the uniform energy bounds:

$$
0 \leqslant E(v) \leqslant E(u)=E_{f}(u)+\log \left(\int_{S^{4}} f e^{4 u} \mathrm{~d} c\right) \leqslant E_{f}\left(u_{0}\right)+\log \left(\max _{S^{4}} f\right)
$$

Using Jensen's inequality we have: $2 \bar{v}:=\int_{S^{4}} 2 v \mathrm{~d} c \leqslant \log \left(\int_{S^{4}} e^{4 v} \mathrm{~d} c\right)=0$. By this, we can obtain the uniform $H^{1}$ norm bound of $v$ for all $t \geqslant 0$ that $\sup _{t}|v(t)|_{H^{1}\left(S^{2}\right)} \leqslant C$. See the proof of Lemma 3.2 in [9]. Using the Aubin-Moser-Trudinger inequality [1] we further have

$$
4 \sup _{\{0 \leqslant t<T\}} \int_{S^{4}}|u(t)| \mathrm{d} c \leqslant \sup _{t} \int_{S^{4}} e^{4|u(t)|} \mathrm{d} c \leqslant C<\infty .
$$

Notice that $v_{t}=u_{t} \circ \phi+\frac{1}{4} e^{-4 v} d i v_{S^{4}}\left(\xi e^{4 v}\right)$ where $\xi=(d \phi)^{-1} \phi_{t}$ is the vector field on $S^{4}$ generating the flow $(\phi(t)), t \geqslant 0$, as in [9], formula (17), with the uniform bound $|\xi|_{L^{\infty}\left(S^{4}\right)}^{2} \leqslant C \int_{S^{4}}|\alpha f-K|^{2} \mathrm{~d} v_{g}$.

With the help of this bound, we can show (see Lemma 3.3 in [9]) that for any $T>0$, the following holds:

$$
\sup _{0 \leqslant t<T} \int_{S^{2}} e^{4|u(t)|} \mathrm{d} c<+\infty
$$

Following the method of A. Malchiodi and M. Struwe [9] (see also Lemma 3.4 in [10]) and using the bound (8) and the growth rate of $\alpha$, we can show that $\int_{S^{4}}|\alpha f-Q|^{2} \mathrm{~d} v_{g} \rightarrow 0$ as $t \rightarrow \infty$. Once getting this curvature decay estimate, we can come to consider the concentration behavior of the metrics $g(t)$. Following [10], we show:

Lemma 2. Let $\left(u_{l}\right)$ be a sequence of smooth functions on $S^{4}$ with associated metrics $g_{l}=e^{2 u_{l}}$ c with $\operatorname{Vol}\left(S^{4}, g_{l}\right)=\frac{8}{3} \pi^{2}, l=1,2, \ldots$ as constructed above. Suppose that there is a smooth non-negative function $Q_{\infty}$, which is positive somewhere on $S^{4}$ such that

$$
\left|Q\left(g_{l}\right)-Q_{\infty}\right|_{L^{2}\left(S^{4}, g_{l}\right)} \rightarrow 0
$$

as $l \rightarrow \infty$. Let $h_{l}=\phi_{l}^{*} g_{l}=e^{2 v_{l}}$ c be defined as before. Then we have either 1) for a subsequence $l \rightarrow \infty$ we have $u_{l} \rightarrow u_{\infty}$ in $H^{4}\left(S^{4}, c\right)$, where $g_{\infty}=e^{2 u_{\infty}}$ c has $Q$-curvature $Q_{\infty}$, or 2 ) there exists a subsequence, still denoted by $\left(u_{l}\right)$ and a point $q \in S^{4}$ with $Q_{\infty}(q)>0$, such that the metric $g_{l}$ has a measure concentration that $\mathrm{d} v_{g_{l}} \rightarrow \frac{8}{3} \pi^{2} \delta_{q}$ weakly in the sense of measures, while $h_{l} \rightarrow c$ in $H^{4}\left(S^{4}, c\right)$ and in particular, $Q\left(h_{l}\right) \rightarrow 3$ in $L^{2}\left(S^{4}\right)$. Moreover, in the latter case the conformal diffeomorphisms $\phi_{l}$ weakly converges in $H^{2}\left(S^{4}\right)$ to the constant map $\phi_{\infty}=q$.

Proof. The case 1) can be proved as Lemma 3.6 in [9]. So we need only prove the case 2). As in [9], we choose $q_{l} \in S^{4}$ and radii $r_{l}>0$ such that

$$
\sup _{q \in S^{4}} \int_{B\left(q, r_{l}\right)}\left|Q\left(g_{l}\right)\right| \mathrm{d} v_{g_{l}} \leqslant \int_{B\left(q_{l}, r_{l}\right)}\left|Q\left(g_{l}\right)\right| \mathrm{d} v_{g_{l}}=2 \pi^{2}
$$

where $B\left(q, r_{l}\right)$ is the geodesic ball in $\left(S^{4}, g_{l}\right)$. Then we have $r_{l} \rightarrow 0$ and we may assume that $q_{l} \rightarrow q$ as $l \rightarrow \infty$. For each $l$, we introduce $\phi_{l}$ as in Lemma 3.6 in [9] so that the functions, $\hat{u}_{l}=u_{l} \circ \phi_{l}+\frac{1}{4} \log \left(\operatorname{det}\left(d \phi_{l}\right)\right)$, satisfy the conformal Q-curvature equation $-P_{R^{4}} \hat{u}_{l}=-\Delta_{R^{4}}^{2} \hat{u}_{l}=2 \hat{Q}_{l} e^{4 \hat{u}_{l}}$, in $R^{4}$, where $\hat{Q}_{l}=Q\left(g_{l}\right) \circ \phi$ and $P_{R^{4}}$ is the Paneitz operator of the standard Euclidean metric $g_{R^{4}}$. Note that for $\hat{g}_{l}=\phi^{*} g_{l}=e^{2 \hat{u}_{l}} g_{R^{4}}$, we have: $\operatorname{Vol}\left(R^{4}, \hat{g}_{l}\right)=\operatorname{Vol}\left(S^{4}, g_{l}\right)=\frac{8}{3} \pi^{2}$. Arguing as in [9], we can conclude a convergent subsequence $\hat{u}_{l} \rightarrow \hat{u}_{\infty}$ in $H_{l o c}^{4}\left(R^{4}\right)$ where $\hat{u}_{\infty}$ satisfies the Liouville type equation, $-\Delta_{R^{4}}^{2} \hat{u}_{\infty}=\hat{Q}_{\infty}(q) e^{4 \hat{u}_{\infty}}$, on $R^{4}$, with the finite volume $\int_{R^{4}} e^{4 \hat{u}} \mathrm{~d} z \leqslant \frac{8}{3} \pi^{2}$.

We need to exclude the case when $Q_{\infty}(q)=0$. If $Q_{\infty}(q)=0$, then $\Delta_{R^{4}} \hat{u}:=\Delta_{R^{4}} \hat{u}_{\infty}$ is a harmonic function in $R^{4}$. Let $\bar{u}(r)$ be the average of $u$ on the circle $\partial B_{r}(0) \subset R^{4}$. Then we have $\Delta_{R^{4}}^{2} \bar{u}=0$. Hence $\Delta_{R^{4}} \bar{u}=A_{0}+B_{0} r^{-2}$ for some constants $A_{0}$ and $B_{0}$, where $r=|x|$. Since $\Delta_{R^{4}} \bar{u}$ is a continuous function on $[0, \infty)$, we have $\Delta_{R^{4}} \bar{u}=A$, which gives us that $\bar{u}=A+B r^{2}+C r^{-2}$, for some constants $A, B$, and $C$. But this is impossible since we have by Jensen's inequality that

$$
2 \pi \int_{0}^{\infty} e^{4 \bar{u}(r)} r^{3} \mathrm{~d} r \leqslant \int_{R^{4}} e^{4 \hat{u}_{\infty}} \mathrm{d} z \leqslant \frac{8}{3} \pi^{2}
$$

The remaining part is the same as in the proof of Lemma 3.6 in [9]. We confer to [9] for the full proof.
With this understanding, we can do the same finite-dimensional dynamics analysis as in Section 5 in [9]. Then arguing as in Section 5 in [9] we can prove Theorem 1. By now the argument is well known, so we omit the detail and refer to [9] for full discussion. Thus, we complete the proof of Theorem 1.

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