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Q-curvature flow with indefinite nonlinearity th

Flot de Q-courbure pour une non-linéarité indéfinie

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ABSTRACT

In this Note, we study Q-curvature flow on S^4 with indefinite nonlinearity. Our result is that the prescribed Q-curvature problem on S^4 has a solution provided the prescribed non-negative Q-curvature f has its positive part, which possesses non-degenerate critical points such that $\Delta_{S^4} f \neq 0$ at the saddle points and an extra condition such as a nontrivial degree counting condition.

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RÉSUMÉ

Dans cette Note on étudie le flot de Q-courbure sur S^4 dans le cas d'une non-linéarité indéfinie. Le résultat montre que le problème de la Q-courbure imposée sur S^4 a une solution à condition que la Q-courbure non négative imposée f ait une partie strictement positive et des points critiques non dégénérés tels que $\Delta_{S^4} f \neq 0$ aux points selles et une condition supplémentaire du type condition non triviale sur le degré.

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1. Introduction

Following the works of A. Chang and P. Yang [5], M. Brendle [4], A. Malchiodi and M. Struwe [9], we study a heat flow method to the prescribed Q-curvature problem on S^4 . Given the Riemannian metric g in the conformal class of standard metric c on S^4 with Q-curvature Q_g . It is well known that

$$Q_g = -\frac{1}{12} \left(\Delta_g R_g - R_g^2 + 3 \left| Rc(g) \right|^2 \right) := Q,$$

where R_g , Rc(g), Δ_g are the scalar curvature, Ricci curvature tensor, the Laplacian operator of the metric g, respectively. Recall the Chern–Gaussian–Bonnet formula on S^4 is,

$$\int_{S^4} Q_g \, \mathrm{d} v_g = 8\pi^2.$$

By this, we know that Q_g has to be positive somewhere. This gives a necessary condition for the prescribed Q-curvature problem on S^4 . Assuming the prescribed curvature function f being positive on S^4 , the heat flow for the Q-curvature problem is a family of metrics of the form $g = e^{2u(x,t)}c$ satisfying:

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$$u_t = \alpha f - Q, \quad x \in S^4, \ t > 0,$$
 (1)

where $u: S^4 \times (0, T) \rightarrow R$ is the unknown, and $\alpha = \alpha(t)$ is defined by

$$\alpha \int_{S^4} f \, \mathrm{d} v_g = 8\pi^2. \tag{2}$$

Here dv_g is the area element with respect to the metric g. It is easy to see that $\alpha_t \int_{S^4} f dv_g = 2\alpha \int_{S^4} (Q - \alpha f) f dv_g$. A. Malchiodi and M. Struwe [9] can show in their Theorem 1.1 that the flow exists globally, furthermore, the flow converges at time infinity provided f is positive and possesses non-degenerate critical points such that $\Delta_{S^4} f \neq 0$ at the saddle points with the condition,

$$\sum_{\{p:\nabla f(p)=0;\,\Delta_{S^4}f(p)<0\}}(-1)^{ind(f,p)}\neq 0.$$

Here $\Delta_{S^4} := \Delta$ is the analyst's Laplacian on the standard 4-sphere (S^4 , c). Recall that $\int_{S^4} dv_c = \frac{8}{3}\pi^2$. The purpose of this Note is to relax their assumption by allowing the function f to have zeros.

Since we have

$$Q = \frac{1}{2}e^{-4u}\left(\Delta^2 u - div\left(\left(\frac{2}{3}R(c)c - 2Rc(c)\right)du\right) + 6\right),$$

Eq. (1) defines a nonlinear parabolic equation for u, and the flow exists at least locally for any initial data $u|_{t=0} = u_0$ and any smooth function f being positive somewhere. Clearly, we have:

$$\partial_t \int\limits_{S^4} \mathrm{d} v_g = 2 \int\limits_{S^4} u_t \, \mathrm{d} v_g = 0$$

We shall assume that the initial data u_0 satisfies the condition,

$$\int_{S^4} f e^{4u} \, \mathrm{d} v_c > 0. \tag{3}$$

We remark that since f can be approximated by positive smooth functions, the set of functions satisfying (3) should be contractible. Then we can use the handle-body theorem following Malchiodi and Struwe [9]. We shall show that the property (3) is preserved along the flow even for f changing signs. In some sense, this may be known to experts. It is easy to compute that

$$Q_t = -4u_t Q - \frac{1}{2} P u_t = 4Q (Q - \alpha f) + P(\alpha f - Q),$$
(4)

where $P = P_g = e^{-4u}P_c$ and P_c is the Paneitz operator in the metric *c* on S^4 [5]. Using (4), we can compute the growth rate of the Calabi-type energy $\int_{S^4} |Q - \alpha f|^2 d\nu_g$.

Our main result is following:

Theorem 1. Let f be a positive somewhere, non-negative smooth function on S^4 with only non-degenerate critical points on the its positive part f_+ with its Morse index $ind(f_+, p)$. Suppose that at each critical point p of f_+ , we have $\Delta f \neq 0$. Let m_i be the number of critical points with f(p) > 0, $\Delta_{S^4} f(p) < 0$ and ind(f, p) = 4 - i. Suppose that there is no solutions with coefficients $k_i \ge 0$ to the system of equations

$$m_0 = 1 + k_0, \qquad m_i = k_{i-1} + k_i, \quad 1 \le i \le 4, \ k_4 = 0.$$

Then f is the Q-curvature of a conformal metric $g = e^{2u}c$ on S^4 .

A similar result for curvature flow to the Nirenberg problem on S^2 has been obtained in [6]. See [2,7,11] and [8] for related.

For simplifying notations, we shall use the conventions that $dc = \frac{dv_c}{\frac{8}{3}\pi^2}$ and $\bar{u} = \bar{u}(t)$ defined by: $\int_{S^4} (u - \bar{u}) dv_c = 0$.

2. Basic properties of the flow

In this section we may allow f to change signs. Recall the following result of Beckner [3]:

$$\int_{S^4} \left(|\Delta u|^2 + 2|\nabla u|^2 + 12u \right) \mathrm{d}c \ge \log \left(\int_{S^4} e^{4u} \, \mathrm{d}c \right) = 0, \tag{5}$$

where $|\nabla u|^2$ is the norm of the gradient of the function u with respect to the standard metric c. Here we have used the fact that $\int_{S^4} e^{4u} dc = 1$ along the flow (1).

We show that the condition (3) is preserved along the flow (1). In fact, letting $E(u) = \int_{S^4} (uPu + 4Q_c u) dc = \int_{S^4} (|\Delta u|_c^2 + 2|\nabla u|_c^2 + 12u) dc$ be the Liouville energy of u and letting, $E_f(u) = E(u) - 3\log(\int_{S^4} fe^{4u} dc)$, be the energy function for the flow (1), we then compute that

$$\partial_t E_f(u) = -\frac{3}{2\pi^2} \int_{S^4} |\alpha f - Q|^2 \, \mathrm{d}\nu_g \leqslant 0.$$
(6)

One may see Lemma 2.1 in [9] for a proof of this formula. Hence

$$E_f(u(t)) \leqslant E_f(u_0), \quad t > 0$$

After using the inequality (5) we have,

$$\log\left(1/\int_{S^4} f e^{4u} \,\mathrm{d}c\right) \leqslant E_f(u_0),\tag{7}$$

which implies that $\int_{S^4} f e^{4u} dv_c > 0$ and furthermore, $e^{E_f(u_0)} \int_{S^4} e^{4u} dc \leqslant \int_{S^4} f e^{4u} dc$.

Note also that $\int_{S^4} f e^{4u} dc = 1/\alpha(t)$. Hence, $\alpha(t) \leq \frac{1}{e^{E_f(u_0)}}$. Using the definition of $\alpha(t)$ we have: $\alpha(t) \geq \frac{1}{\max_{S^4} f}$. We then conclude that $\alpha(t)$ is uniformly bounded along the flow, i.e.,

$$\frac{1}{\max_{S^4} f} \leqslant \alpha(t) \leqslant \frac{1}{e^{E_f(u_0)}}.$$
(8)

We shall use this inequality to replace (26) in [9] in the study of the normalized flow, which will be defined in the next section following the work of A. Malchiodi and M. Struwe [9]. If we have a global Q-curvature flow, then using (6) we have:

$$2\int_{0}^{\infty} dt \int_{S^4} |\alpha f - Q|^2 d\nu_g \leq 4\pi \Big(E_f(u_0) + \log \max_{S^4} f \Big).$$

Hence we have a suitable sequence $t_l \to \infty$ with associated metrics $g_l = g(t_l)$ and $\alpha(t_l) \to \alpha > 0$, and letting $Q_l = Q(g_l)$ be the Q-curvature of the metric g_l , such that $\int_{S^4} |Q_l - \alpha f|^2 \to 0$ ($t_l \to \infty$). Therefore, once we have a limiting metric g_{∞} of the sequence of the metrics g_l , it follows that $Q(g_{\infty}) = \alpha f$. After a re-scaling, we see that f is the Q-curvature of the metric βg_{∞} for some $\beta > 0$, which implies our Theorem 1.

3. Normalized flow and the proof of Theorem 1

In this section, we fix f assumed in Theorem 1. We now introduce the normalized flow. For the given flow $g(t) = e^{2u(t)}c$ on S^4 , there exists a family of conformal diffeomorphisms $\phi = \phi(t) : S^4 \to S^4$, which depends smoothly on the time variable t, such that for the metrics $h = \phi^*g$, we have:

$$\int_{S^4} x \, \mathrm{d} v_h = 0, \quad \text{for all } t \ge 0.$$

Here $x = (x^1, x^2, x^3, x^4, x^5) \in S^4 \subset R^5$ is a position vector of the standard 4-sphere. Let $v = u \circ \phi + \frac{1}{4} \log(\det(d\phi))$. Then we have $h = e^{2v}c$. Using the conformal invariance of the Liouville energy [5], we have: E(v) = E(u), and furthermore, $Vol(S^4, h) = Vol(S^4, g) = \frac{8}{3}\pi^2$, for all $t \ge 0$.

Assume u(t) satisfies (1) and (2). Then we have the uniform energy bounds:

$$0 \leqslant E(v) \leqslant E(u) = E_f(u) + \log\left(\int_{S^4} f e^{4u} dc\right) \leqslant E_f(u_0) + \log\left(\max_{S^4} f\right).$$

Using Jensen's inequality we have: $2\bar{v} := \int_{S^4} 2v \, dc \leq \log(\int_{S^4} e^{4v} \, dc) = 0$. By this, we can obtain the uniform H^1 norm bound of v for all $t \geq 0$ that $\sup_t |v(t)|_{H^1(S^2)} \leq C$. See the proof of Lemma 3.2 in [9]. Using the Aubin–Moser–Trudinger inequality [1] we further have

$$4\sup_{\{0\leqslant t< T\}}\int_{S^4} |u(t)| \,\mathrm{d} c\leqslant \sup_t \int_{S^4} e^{4|u(t)|} \,\mathrm{d} c\leqslant C<\infty.$$

Notice that $v_t = u_t \circ \phi + \frac{1}{4} e^{-4v} div_{S^4}(\xi e^{4v})$ where $\xi = (d\phi)^{-1}\phi_t$ is the vector field on S^4 generating the flow $(\phi(t)), t \ge 0$, as in [9], formula (17), with the uniform bound $|\xi|^2_{L^{\infty}(S^4)} \le C \int_{S^4} |\alpha f - K|^2 dv_g$.

With the help of this bound, we can show (see Lemma 3.3 in [9]) that for any T > 0, the following holds:

$$\sup_{0\leqslant t< T}\int_{S^2}e^{4|u(t)|}\,\mathrm{d} c<+\infty.$$

Following the method of A. Malchiodi and M. Struwe [9] (see also Lemma 3.4 in [10]) and using the bound (8) and the growth rate of α , we can show that $\int_{S^4} |\alpha f - Q|^2 dv_g \to 0$ as $t \to \infty$. Once getting this curvature decay estimate, we can come to consider the concentration behavior of the metrics g(t). Following [10], we show:

Lemma 2. Let (u_l) be a sequence of smooth functions on S^4 with associated metrics $g_l = e^{2u_l}c$ with $Vol(S^4, g_l) = \frac{8}{3}\pi^2$, l = 1, 2, ... as constructed above. Suppose that there is a smooth non-negative function Q_{∞} , which is positive somewhere on S^4 such that

$$\left| Q\left(g_{l}\right) - Q_{\infty} \right|_{L^{2}(S^{4},g_{l})} \to 0$$

as $l \to \infty$. Let $h_l = \phi_l^* g_l = e^{2v_l} c$ be defined as before. Then we have either 1) for a subsequence $l \to \infty$ we have $u_l \to u_\infty$ in $H^4(S^4, c)$, where $g_\infty = e^{2u_\infty} c$ has Q-curvature Q_∞ , or 2) there exists a subsequence, still denoted by (u_l) and a point $q \in S^4$ with $Q_\infty(q) > 0$, such that the metric g_l has a measure concentration that $dv_{g_l} \to \frac{8}{3}\pi^2\delta_q$ weakly in the sense of measures, while $h_l \to c$ in $H^4(S^4, c)$ and in particular, $Q(h_l) \to 3$ in $L^2(S^4)$. Moreover, in the latter case the conformal diffeomorphisms ϕ_l weakly converges in $H^2(S^4)$ to the constant map $\phi_\infty = q$.

Proof. The case 1) can be proved as Lemma 3.6 in [9]. So we need only prove the case 2). As in [9], we choose $q_l \in S^4$ and radii $r_l > 0$ such that

$$\sup_{q\in S^4}\int_{B(q,r_l)} \left| Q\left(g_l\right) \right| \mathrm{d}\nu_{g_l} \leqslant \int_{B(q_l,r_l)} \left| Q\left(g_l\right) \right| \mathrm{d}\nu_{g_l} = 2\pi^2,$$

where $B(q, r_l)$ is the geodesic ball in (S^4, g_l) . Then we have $r_l \to 0$ and we may assume that $q_l \to q$ as $l \to \infty$. For each l, we introduce ϕ_l as in Lemma 3.6 in [9] so that the functions, $\hat{u}_l = u_l \circ \phi_l + \frac{1}{4} \log(\det(d\phi_l))$, satisfy the conformal Q-curvature equation $-P_{R^4}\hat{u}_l = -\Delta_{R^4}^2\hat{u}_l = 2\hat{Q}_l e^{4\hat{u}_l}$, in R^4 , where $\hat{Q}_l = Q(g_l) \circ \phi$ and P_{R^4} is the Paneitz operator of the standard Euclidean metric g_{R^4} . Note that for $\hat{g}_l = \phi^* g_l = e^{2\hat{u}_l} g_{R^4}$, we have: $Vol(R^4, \hat{g}_l) = Vol(S^4, g_l) = \frac{8}{3}\pi^2$. Arguing as in [9], we can conclude a convergent subsequence $\hat{u}_l \to \hat{u}_\infty$ in $H^4_{loc}(R^4)$ where \hat{u}_∞ satisfies the Liouville type equation, $-\Delta_{R^4}^2 \hat{u}_\infty = \hat{Q}_\infty(q)e^{4\hat{u}_\infty}$, on R^4 , with the finite volume $\int_{R^4} e^{4\hat{u}_\infty} dz \leq \frac{8}{3}\pi^2$.

We need to exclude the case when $Q_{\infty}(q) = 0$. If $Q_{\infty}(q) = 0$, then $\Delta_{R^4}\hat{u} := \Delta_{R^4}\hat{u}_{\infty}$ is a harmonic function in R^4 . Let $\bar{u}(r)$ be the average of u on the circle $\partial B_r(0) \subset R^4$. Then we have $\Delta_{R^4}^2 \bar{u} = 0$. Hence $\Delta_{R^4} \bar{u} = A_0 + B_0 r^{-2}$ for some constants A_0 and B_0 , where r = |x|. Since $\Delta_{R^4} \bar{u}$ is a continuous function on $[0, \infty)$, we have $\Delta_{R^4} \bar{u} = A$, which gives us that $\bar{u} = A + Br^2 + Cr^{-2}$, for some constants A, B, and C. But this is impossible since we have by Jensen's inequality that

$$2\pi \int_{0}^{\infty} e^{4\bar{u}(r)} r^3 \, \mathrm{d}r \leqslant \int_{R^4} e^{4\hat{u}_{\infty}} \, \mathrm{d}z \leqslant \frac{8}{3}\pi^2.$$

The remaining part is the same as in the proof of Lemma 3.6 in [9]. We confer to [9] for the full proof. \Box

With this understanding, we can do the same finite-dimensional dynamics analysis as in Section 5 in [9]. Then arguing as in Section 5 in [9] we can prove Theorem 1. By now the argument is well known, so we omit the detail and refer to [9] for full discussion. Thus, we complete the proof of Theorem 1.

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