The Ghirlanda–Guerra identities for mixed $p$-spin model

Les identités de Ghirlanda–Guerra pour les mélanges de modèles à $p$-spin

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1. Introduction and main result

A generic mixed $p$-spin Hamiltonian $H_N(\sigma)$ indexed by spin configurations $\sigma \in \{-1, +1\}^N$ is defined as a linear combination

$$H_N(\sigma) = \sum_{p \geq 1} \beta_p H_p(\sigma)$$

of $p$-spin Sherrington–Kirkpatrick Hamiltonians

$$H_p(\sigma) = \frac{1}{N(p-1)/2} \sum_{1 \leq i_1, \ldots, i_p \leq N} g_{i_1, \ldots, i_p} \sigma_{i_1} \cdots \sigma_{i_p}$$

where $(g_{i_1, \ldots, i_p})$ are i.i.d. standard Gaussian random variables, also independent for all $p \geq 1$. For simplicity of notation, we will keep the dependence of $H_p$ on $N$ implicit. If a model involves an external field parameter $h \in \mathbb{R}$ then the (random) Gibbs measure on $\{-1, +1\}^N$ corresponding to the Hamiltonian $H_N$ is defined by

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\[ G_N(\sigma) = \frac{1}{Z_N} \exp \left( H_N(\sigma) + h \sum_{i \in N} \sigma_i \right) \]  

(3)

where \( Z_N \) is the normalizing factor called the partition function. As usual, we will denote by \( \langle \cdot \rangle \) the expectation with respect to the product Gibbs measure \( G_N^{\otimes \infty} \). One of the most important properties of the Gibbs measure \( G_N \) was discovered by Ghirlanda and Guerra in [2] who showed that on average over some small perturbation of the parameters \( (\beta_p)_{p \geq 2} \) the annealed product Gibbs measure satisfies a family of distributional identities which are now called the Ghirlanda–Guerra identities. A more convenient version of this result proved in [7] can be formulated as follows. There exists a small perturbation of parameters was proved for

\[ \lim_{N \to \infty} \left| \frac{1}{N} \mathbb{E}[f(R_{1,n+1})] - \frac{1}{n} \mathbb{E}(f(R_{1,2}^p) - \frac{1}{n} \sum_{l=2}^n \mathbb{E}[f(R_{1,l})] \right| = 0, \]  

(4)

where \( \langle \cdot \rangle \) is now the Gibbs average corresponding to the Hamiltonian (1) with perturbed parameters \( (\beta_{N,p}) \) and \( R_{1,p} = N^{-1} \sum_{i \leq N} \sigma_i \sigma_i^l \) is the overlap of configurations \( \sigma^l \) and \( \sigma^l \). Of course, the ultimate goal would be to show that (4) holds without perturbing the parameters \( (\beta_p) \) which would mean that the joint distribution of the overlaps \( (R_{1,p})_{p \geq 1} \) under the annealed product Gibbs measure \( G_N^{\otimes \infty} \) asymptotically satisfies the following distributional identities (up to symmetry considerations): for any \( n \geq 2 \), conditionally on \( (R_{1,p})_{1 \leq p \leq n} \) the law of \( R_{1,n+1} \) is given by the mixture \( n^{-1} \mu_n + n^{-1} \sum_{i=2}^n \delta_{R_{1,i}} \) where \( \mu_n \) is the law of \( R_{1,2} \). Toward this goal, a stronger version of (4) for the original Hamiltonian (1) without any perturbation of parameters was proved for \( p = 1 \) in [1] under the additional assumption that \( \beta_1 \neq 0 \) and a non-restrictive assumption on the limit of the free energy \( F_N = N^{-1} \mathbb{E} \log Z_N \). Here we will prove the same result for all \( p \) under the assumptions and as a direct consequence of the seminal work of Talagrand in [6] where the validity of the Parisi formula was proved. Namely, from now on we will assume that the sum in (1) is taken over \( p = 1 \) and even \( p \geq 2 \). In this case, it was proved in [6] that the limit of the free energy

\[ \lim_{N \to \infty} F_N(\beta) = P(\beta) \]  

(5)

exists and is given by the Parisi formula \( P(\beta) \) discovered in [4]. The exact definition of \( P(\beta) \) will not be important to us and the only nontrivial property that we will use is its differentiability in each coordinate \( \beta_p \) which was proved in [5] (see also [3]).

**Theorem 1.1.** For \( p = 1 \) and for all even \( p \geq 2 \),

\[ \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \left[ \left| H_p(\sigma) - \mathbb{E}(H_p(\sigma)) \right| \right] = 0. \]  

(6)

If \( \beta_p \neq 0 \) then (6) implies (4) by the usual integration by parts. In particular, if \( \beta_p \neq 0 \) for \( p = 1 \) and all even \( p \geq 2 \), the positivity principle of Talagrand proved in [8] implies the strong version of the extended Ghirlanda–Guerra identities without any perturbation of the parameters.

**Remark.** We will see that the proof does not depend on the specific form of the Hamiltonian (1) and the result can be formulated in more generality. Namely, given a sequence of random measures \( \nu_N \) on some measurable space \((\Sigma, S)\) and a sequence of random processes \( A_N \) indexed by \( \sigma \in \Sigma \), consider a sequence of Gibbs’ measures \( G_N \) defined by the change of density

\[ dG_N(\sigma) = Z_N^{-1} \exp(xA_N(\sigma)) \, d\nu_N(\sigma). \]

Let \( \psi_N(x) = N^{-1} \log Z_N \) and \( F_N(x) = \mathbb{E} \psi_N(x) \). Suppose that \( \mathbb{E} |\psi_N(x) - F_N(x)| \to 0 \) and \( F_N(x) \to P(x) \) in some neighborhood of \( x_0 \), and suppose that the limit \( P(x) \) is differentiable at \( x_0 \). Then

\[ \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \left[ \left| A_N(\sigma) - \mathbb{E}(A_N(\sigma)) \right| \right]_{x_0} = 0, \]  

(7)

assuming some measurability and integrability conditions on \( A_N \) and \( \nu_N \) which will be clear from the proof and are usually trivially satisfied. In Theorem 1.1 we simply appeal to the results in [5] and [6].

**Proof of Theorem 1.1.** It has been observed for a long time that (see, for example, [1])

\[ \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \left[ \left| H_p(\sigma) - \mathbb{E}(H_p(\sigma)) \right| \right] = 0. \]
This is where one uses the fact that $\mathbb{E}|\psi_N - F_N| \to 0$, which is well known for $p$-spin models. It remains to prove that

$$\lim_{N \to \infty} \frac{1}{N} \mathbb{E}[|H_p(\sigma) - \langle H_p(\sigma) \rangle|] = 0. \quad (8)$$

This was proved in [1] for $p = 1$, but here we will show how this can be obtained as a direct consequence of (5) for all $p$. Let $\langle \cdot \rangle_x$ denote the Gibbs average corresponding to the Hamiltonian (1) where $\beta_p$ has been replaced by $x$. Consider $\beta_p^* > \beta_p$ and let $\delta = \beta_p^* - \beta_p$. We start with the following obvious equation,

$$\int_{\beta_p}^{\beta_p^*} \mathbb{E}[|H_p(\sigma^1) - H_p(\sigma^2)|]_x \, dx = \delta \mathbb{E}[|H_p(\sigma^1) - H_p(\sigma^2)|]_{\beta_p} + \int_{\beta_p}^{\beta_p^*} \frac{\partial}{\partial t} \mathbb{E}[|H_p(\sigma^1) - H_p(\sigma^2)|]_{t} \, dt \, dx. \quad (9)$$

Since

$$\left| \frac{\partial}{\partial t} \mathbb{E}[|H_p(\sigma^1) - H_p(\sigma^2)|]_{t} \right| = \mathbb{E}[|H_p(\sigma^1) - H_p(\sigma^2)|((H_p(\sigma^1) + H_p(\sigma^2) - 2H_p(\sigma^2))]_t \leq 2\mathbb{E}[(H_p(\sigma^1) - H_p(\sigma^2))^2]_t \leq 8\mathbb{E}[(H_p(\sigma) - \langle H_p(\sigma) \rangle^2]_t$$

Eq. (9) implies

$$\mathbb{E}[|H_p(\sigma^1) - H_p(\sigma^2)|]_{\beta_p} \leq \frac{1}{\delta} \int_{\beta_p}^{\beta_p^*} \mathbb{E}[|H_p(\sigma^1) - H_p(\sigma^2)|]_x \, dx + \frac{8}{\delta} \int_{\beta_p}^{\beta_p^*} \mathbb{E}[(H_p(\sigma) - \langle H_p(\sigma) \rangle^2]_t \, dt \, dx$$

$$\leq \frac{2}{\delta} \int_{\beta_p}^{\beta_p^*} \mathbb{E}[|H_p(\sigma) - \langle H_p(\sigma) \rangle|]_x \, dx + 8 \int_{\beta_p}^{\beta_p^*} \mathbb{E}[(H_p(\sigma) - \langle H_p(\sigma) \rangle^2]_t \, dt$$

$$\leq 2 \left( \frac{1}{\delta} \int_{\beta_p}^{\beta_p^*} \mathbb{E}[(H_p(\sigma) - \langle H_p(\sigma) \rangle^2]_x \, dx \right)^{1/2} + 8 \int_{\beta_p}^{\beta_p^*} \mathbb{E}[(H_p(\sigma) - \langle H_p(\sigma) \rangle^2]_t \, dx.$$

Therefore, if we denote

$$\Delta_N = \frac{1}{N} \int_{\beta_p}^{\beta_p^*} \mathbb{E}[(H_p(\sigma) - \langle H_p(\sigma) \rangle^2]_x \, dx$$

we showed that

$$\frac{1}{N} \mathbb{E}[|H_p(\sigma) - \langle H_p(\sigma) \rangle|] \leq \frac{1}{N} \mathbb{E}[|H_p(\sigma^1) - H_p(\sigma^2)|] \leq 2\sqrt{\frac{\Delta_N}{N\delta}} + 8\Delta_N. \quad (10)$$

If for a moment we think of $F_N = F_N(x)$ as a function of $x$ only then

$$F'_N(x) = \frac{1}{N} \mathbb{E}[H_p(\sigma)]_x \quad \text{and} \quad F''_N(x) = \frac{1}{N} \mathbb{E}[(H_p(\sigma) - \langle H_p(\sigma) \rangle^2]_x$$

so that $\Delta_N = F'_N(\beta_p) - F'_N(\beta_p^*)$. Since $F_N(x)$ is convex, for any $\gamma > 0$,

$$\Delta_N = F'_N(\beta_p) - F'_N(\beta_p) \leq \frac{F_N(\beta_p^* + \gamma) - F_N(\beta_p^*) - F_N(\beta_p) - F_N(\beta_p - \gamma)}{\gamma}$$

and, therefore, Eqs. (10) and (5) now imply

$$\lim_{N \to \infty} \frac{1}{N} \mathbb{E}[|H_p(\sigma) - \langle H_p(\sigma) \rangle|] \leq 8 \left( \frac{P(\beta_p^* + \gamma) - P(\beta_p^*)}{\gamma} - \frac{P(\beta_p) - P(\beta_p - \gamma)}{\gamma} \right)$$

where again we write $P = P(x)$ as a function of $x$ only. Letting $\beta_p^* \to \beta_p$ first and then letting $\gamma \to 0$ and using that $P(x)$ is differentiable proves the result. $\Box$
References

[8] M. Talagrand, Mean field models for spin glasses, manuscript.