

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I



www.sciencedirect.com

Mathematical Problems in Mechanics

Exact controllability of Naghdi shells

Contrôlabilité exacte d'une coque de Naghdi

Bernadette Miara^a, Gustavo Perla^{b,c}

^a Université Paris-Est, École supérieure d'ingénieurs en électrotechnique et électronique, département de modélisation et simulation numérique, 2, boulevard Blaise-Pascal, 93160 Noisy-le-Grand, France

^b National Laboratory of Scientific Computation LNCC/MCT, Rua Getulio Vargas 333, Quitandinha, Petropolis 25651-070, RJ, Brasil

^c Institute of Mathematics, Federal University of Rio de Janeiro, RJ, P.O. 68530, Rio de Janeiro, RJ, Brasil

ARTICLE INFO

Article history: Received 15 November 2009 Accepted 30 December 2009 Available online 19 February 2010

Presented by Philippe G. Ciarlet

ABSTRACT

We consider the second order evolution equations of the displacement and rotation fields in any points of the middle surface of a shell in Naghdi's formulation. We establish that, with two controls, the exact controllability on part of the lateral boundary is achieved under the sufficient geometrical condition that the middle surface of the shell is "not too far" from a plane in a sense that is made more precise in the proof.

© 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

RÉSUMÉ

Nous considérons les équations d'évolution qui régissent la dynamique d'une coque mince dans la théorie de Naghdi. Nous établissons la contrôlabilité exacte grâce à deux contrôles qui agissent sur une partie de la frontière latérale de la coque sous la condition géométrique suffisante que sa surface moyenne ne s'éloigne « pas trop » d'un plan dans un sens qui est précisé dans la preuve.

© 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Version française abrégée

On considère le domaine $\omega \subset \mathbb{R}^2$ et l'injection $s : \overline{\omega} \to \mathbb{R}^3$ qui définissent la surface moyenne $s(\overline{\omega})$ de la coque. Dans le cadre du modèle de Naghdi, le champ de déplacement tridimensionnel U est de la forme (1), où $u = (u_1, u_2, u_3)$ et $\theta = (\theta_1, \theta_2)$ représentent les composantes covariantes du déplacement et la rotation en chaque point de la surface moyenne. Le couple (u, θ) est donné par la minimisation de l'énergie (2) (pour un résultat d'existence voir [1]) qu'on peut écrire aussi sous la forme du système (4). Le couplage entre le déplacement et la rotation est mesuré par le coefficient k qui apparait dans l'opérateur auto-adjoint \mathcal{A} . La relation entre les opérateurs \mathcal{A} , \mathcal{A} , \mathcal{B} est explicitée par la formule de Green (5).¹

On s'intéresse dans cette Note à la contrôlabilité exacte du système couplé décrit par le problème d'évolution (6),

 $\begin{cases} \mathbf{y}'' + \mathbf{A}(\mathbf{y}, \boldsymbol{\zeta}) = \mathbf{0} & \text{dans } Q = \omega \times (0, T), \\ \boldsymbol{\zeta}'' + \mathbf{B}(\mathbf{y}, \boldsymbol{\zeta}) = \mathbf{0} & \text{dans } Q, \\ \mathbf{y} = \hat{\mathbf{y}}, \quad \boldsymbol{\zeta} = \hat{\boldsymbol{\zeta}} & \text{sur } \boldsymbol{\Sigma} = \partial \omega \times (0, T), \\ \mathbf{y}(0) = \mathbf{y}^0, \quad \mathbf{y}'(0) = \mathbf{y}^1, \quad \boldsymbol{\zeta}(0) = \boldsymbol{\zeta}^0, \quad \boldsymbol{\zeta}'(0) = \boldsymbol{\zeta}^1 & \text{dans } \omega. \end{cases}$

E-mail addresses: bernadette.miara@univ-paris-est.fr (B. Miara), perla@lncc.br (G. Perla).

¹ Toute les quantités introduites dans cette Note sont complètement détaillées dans [6].

¹⁶³¹⁻⁰⁷³X/\$ – see front matter © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. doi:10.1016/j.crma.2009.12.019

Dans le Théorème 1 de la section 2 on établit les principales propriétés du système homogène associé à (6). En particulier, après avoir défini l'énergie E(t) dans (9) on montre que pour tout multiplicateur $\mathbf{q} \in \mathbf{C}(\overline{\omega})$ on a l'inégalité directe (10) et l'identité (11). On établit en particulier la condition d'observabilité (12) suivante :

Il existe deux constantes positives $\tilde{K}(\omega, \boldsymbol{s}, k)$, $\tilde{T}(\omega, \boldsymbol{s}, k)$ qui dépendent du domaine ω , de l'immersion \boldsymbol{s} et du coefficient de couplage k, telles que : $\frac{1}{2} \int_{\Sigma} (\boldsymbol{p} \cdot \boldsymbol{n}) \mathcal{A}(\boldsymbol{u}, \boldsymbol{\theta}; \boldsymbol{u}, \boldsymbol{\theta}) \sqrt{a} \, d\omega \, dt \ge (1 - \tilde{K}(\omega, \boldsymbol{s}, k))(T - \tilde{T}(\omega, \boldsymbol{s}, k))E^0$ avec le multiplicateur $\boldsymbol{p} = x - x^0$.

Dans la section 3 on en déduit la contrôlabilité exacte. On introduit la partition $\partial \omega = \partial \omega^+ \cup \partial \omega^-$ de la frontière telle que $\mathbf{p} \cdot \mathbf{n} > 0$ sur $\partial \omega^+$ et $\mathbf{p} \cdot \mathbf{n} < 0$ sur $\partial \omega^-$. On pose $\mathcal{H} = \mathbf{H}_0^1(\omega) \times \mathbf{L}^2(\omega)$ et $\mathcal{H}' = \mathbf{L}^2(\omega) \times \mathbf{H}^{-1}(\omega)$.

Théorème 2. On suppose que les conditions initiales satisfont la régularité $(\mathbf{y}^0, \mathbf{y}^1) \in \mathcal{H}', (\boldsymbol{\zeta}^0, \boldsymbol{\zeta}^1) \in \mathcal{H}'$. Alors, si $\tilde{K}(\omega, \mathbf{s}, \mathbf{k}) < 1$, il existe un temps minimal $T^0 = T^0(\omega, \mathbf{s}) > 0$ et un contrôle frontière $(\hat{\mathbf{y}}, \hat{\mathbf{\zeta}}) \in \mathbf{L}^2(\Sigma^+) \times \mathbf{L}^2(\Sigma^+)$ donné par $\hat{\mathbf{y}} = \partial_n \mathbf{u}, \hat{\mathbf{\zeta}} = \partial_n \theta$ sur $\Sigma^+ = \partial \omega^+ \times (0, T)$, où (\mathbf{u}, θ) est la solution unique du problème d'évolution homogène (7) associé aux conditions initiales $(\mathbf{u}^0, \mathbf{u}^1) \in \mathcal{H}, (\theta^0, \theta^1) \in \mathcal{H}$. Ce contrôle permet d'obtenir $\mathbf{y}(T) = \mathbf{y}'(T) = \mathbf{0}, \mathbf{\zeta}(T) = \mathbf{\zeta}'(T) = \mathbf{0}$ dans ω pour tout $T \ge T^0$.

La preuve s'inspire de celle donnée dans [4]. La condition $\tilde{K}(\omega, \mathbf{s}, k) < 1$ exprime que la surface moyenne de la coque est «peu éloignée» d'un plan. La preuve que T^0 ne dépend plus du couplage k est obtenue en adaptant les résultats de Komornik pour les plaques de Mindlin–Reissner [2,3]. Un calcul explicite de \tilde{K} dans le cas d'une coque cylindrique se trouve dans [6].

1. Introduction

Let us consider a bounded, connected subset $\omega \subset \mathbb{R}^2$ (of curvilinear coordinates) with regular boundary $\partial \omega$ and an injective, smooth enough mapping $s: \overline{\omega} \to \mathbb{R}^3$ which defines the middle surface $s(\overline{\omega})$ of a shell. If we denote $s^{\varepsilon}(x_1, x_2, x_3) = s(x_1, x_2) + \varepsilon x_3 a_3(x_1, x_2)$ with $(x_1, x_2) \in \overline{\omega}, x_3 \in (-1, 1)$ and a_3 the unit normal to each point of the middle surface of the shell under study, then the reference configuration of the shell with thickness 2ε is the domain $s^{\varepsilon}(\overline{\omega} \times (-\varepsilon, \varepsilon))$ which we suppose to be stress-free. Subjected to applied forces (f, g), the elastic body which occupies the set $s^{\varepsilon}(\overline{\omega} \times (-\varepsilon, \varepsilon))$ undergoes a three-dimensional displacement field² $\mathbf{U} = (U_1, U_2, U_3) : \overline{\omega} \to \mathbb{R}^3$ which, in the framework of Naghdi's theory, takes the form:

$$\begin{array}{l} U_1(x_1, x_2, x_3) = u_1(x_1, x_2) + \varepsilon x_3 \theta_1(x_1, x_2), \\ U_2(x_1, x_2, x_3) = u_2(x_1, x_2) + \varepsilon x_3 \theta_2(x_1, x_2), \\ U_3(x_1, x_2, x_3) = u_3(x_1, x_2), \end{array}$$
(1)

with $(x_1, x_2) \in \overline{\omega}, x_3 \in (-1, 1)$ and where the 2-dimensional field $\boldsymbol{u} = (u_1, u_2, u_3) : \overline{\omega} \to \mathbb{R}^3$ represents the covariant components of the displacement of any point of the mid-surface of the shell and the 2-dimensional field $\boldsymbol{\theta} = (\theta_1, \theta_2) : \overline{\omega} \to \mathbb{R}^2$ represents the rotation of the normal to the mid-surface of the shell at the same point. The couple $(\boldsymbol{u}, \boldsymbol{\theta})$ is given as the minimum (in an appropriate functional space that is described later) of the static energy

$$\frac{1}{2} \int_{\omega} \mathcal{A}(\boldsymbol{v}, \boldsymbol{\psi}; \boldsymbol{v}, \boldsymbol{\psi}) \sqrt{a} \, \mathrm{d}x - \int_{\omega} (\boldsymbol{f} \cdot \boldsymbol{v} + \boldsymbol{g} \cdot \boldsymbol{\psi}) \sqrt{a} \, \mathrm{d}x,$$
(2)

where the elasticity symmetric auto-adjoint operator A reads

$$\mathcal{A}(\boldsymbol{u},\boldsymbol{\theta};\boldsymbol{v},\boldsymbol{\psi}) = \boldsymbol{\varepsilon} a^{\alpha\beta\sigma\tau} \gamma_{\alpha\beta}(\boldsymbol{u}) \gamma_{\sigma\tau}(\boldsymbol{v}) + \boldsymbol{\varepsilon} k a^{\alpha\beta3} \gamma_{\alpha3}(\boldsymbol{u},\boldsymbol{\theta}) \gamma_{\beta3}(\boldsymbol{v},\boldsymbol{\psi}) + \frac{\boldsymbol{\varepsilon}^3}{3} a^{\alpha\beta\sigma\tau} \rho_{\alpha\beta}(\boldsymbol{u},\boldsymbol{\theta}) \rho_{\sigma\tau}(\boldsymbol{v},\boldsymbol{\psi}). \tag{3}$$

In the case of a Saint Venant–Kirchhoff elastic material, characterized by its two Lamé constants (λ, μ) the fourth order 2-dimensional elasticity tensor reads $a^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})$ and the second order tensor $a^{\alpha\beta3} = 2\mu a^{\alpha\beta}$ where $a^{\alpha\beta}$ represent the contravariant components of the metric tensor of the shell and $a = \det(a^{\alpha\beta})^{-1}$. The others quantities are the shear correction coupling factor *k* and the 2-dimensional change of metric and change of curvature tensors γ and *r* (for a complete definition see [6]):

$$\begin{cases} \gamma_{\alpha\beta}(\mathbf{v}) = 1/2(\partial_{\alpha}v_{\beta} + \partial_{\beta}v_{\alpha}) - \Gamma_{\alpha\beta}^{\sigma}v_{\sigma} - b_{\alpha\beta}v_{3}, \\ \gamma_{\alpha3}(\mathbf{v}, \psi) = 1/2(\partial_{\alpha}v_{3} + b_{\alpha}^{\sigma}v_{\sigma} + \psi_{\alpha}), \\ r_{\alpha\beta}(\mathbf{v}, \psi) = 1/2(b_{\alpha}^{\sigma}\partial_{\beta}v_{\sigma} + b_{\beta}^{\sigma}\partial_{\alpha}v_{\sigma}) - 1/2v_{\tau}(b_{\alpha}^{\sigma}\Gamma_{\beta\sigma}^{\tau} + b_{\beta}^{\sigma}\Gamma_{\alpha\sigma}^{\tau}) - b_{\alpha}^{\sigma}b_{\beta\sigma}v_{3} - 1/2(\partial_{\beta}\psi_{\alpha} + \partial_{\alpha}\psi_{\beta}) + \Gamma_{\alpha\beta}^{\sigma}\psi_{\sigma}. \end{cases}$$

We note that the three components of the displacement field are coupled with the two components of the rotation θ . In the sequel we assume that the shell is regular enough so that all the quantities such as $(a^{\alpha\beta}, \Gamma^{\sigma}_{\alpha\beta}, b_{\alpha\beta}, ...)$ have as many derivatives as necessary.

² Latin exponents and indices take their values in the set {1; 2; 3}, Greek exponents and indices take their values in the set {1; 2}, Einstein convention for repeated exponents and indices is used and bold face letters represent vectors or vector spaces. Elastic displacements (u, v, ...) will be represented with Latin letters, rotation $(\theta, \psi, ...)$ with Greek letters.

We denote $\partial \omega_0$ a part of the boundary of $\partial \omega$ with positive length and the space $\mathbf{V}_0 = \{(\mathbf{v}, \boldsymbol{\psi}) \in (H^1(\omega))^3 \times (H^1(\omega))^2, v_i = \psi_\alpha = 0 \text{ on } \partial \omega_0\}$. Let us recall that, for all $\mathbf{f} \in \mathbf{L}^2(\omega)$ and $\mathbf{g} \in \mathbf{L}^2(\omega)$, there exists a unique weak solution $(\mathbf{u}, \theta) \in \mathbf{V}_0$ to the variational system $\int_{\omega} \mathcal{A}(\mathbf{u}, \theta; \mathbf{v}, \boldsymbol{\psi}) \sqrt{a} dx = \int_{\omega} (\mathbf{f} \cdot \mathbf{v} + \mathbf{g} \cdot \boldsymbol{\psi}) \sqrt{a} dx$ for all $(\mathbf{v}, \boldsymbol{\psi}) \in \mathbf{V}_0$ (for a proof see [1]).

Therefore the equilibrium equations³ associated to the energy (2) take the form of the system

$$\begin{cases} A(u,\theta) + f = \mathbf{0} & \text{in } \omega, \\ B(u,\theta) + g = \mathbf{0} & \text{in } \omega, \end{cases}$$
(4)

where the operators $(\mathcal{A}, \mathbf{A}, \mathbf{B})$ and the boundary operators (\mathbf{X}, \mathbf{Y}) satisfy Green's formula⁴ for any classical solution (\mathbf{u}, θ) of (4)

$$\int_{\omega} \left(\mathbf{A}(\mathbf{u},\boldsymbol{\theta}) \cdot \mathbf{v} + \mathbf{B}(\mathbf{u},\boldsymbol{\theta}) \cdot \boldsymbol{\psi} \right) \sqrt{a} \, \mathrm{d}x = \int_{\omega} \mathcal{A}(\mathbf{u},\boldsymbol{\theta};\mathbf{v},\boldsymbol{\psi}) \sqrt{a} \, \mathrm{d}x - \int_{\partial \omega} \left(\mathbf{X}(\mathbf{u},\boldsymbol{\theta},\mathbf{n}) \cdot \mathbf{v} + \mathbf{Y}(\mathbf{u},\boldsymbol{\theta},\mathbf{n}) \cdot \boldsymbol{\psi} \right) \sqrt{a} \, \mathrm{d}\omega, \tag{5}$$

where **n** is the outer unit normal to $\partial \omega$. For any positive time T > 0 we can now formulate the evolution problem posed in the domain $Q = \omega \times (0, T)$. Let us assume that the body is subjected to two Dirichlet conditions $(\hat{y}, \hat{\xi}) : \Sigma \to \mathbb{R}^3 \times \mathbb{R}^2$, applied on the boundary $\Sigma = \partial \omega \times (0, T)$ and Cauchy initial conditions $(y^0, y^1, \xi^0, \xi^1) : \omega \to (\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2)$. Hence the evolution problem with Dirichlet boundary conditions and Cauchy initial conditions reads (without any loss of generality we fix to value 1 the mass density of the material):

$$\begin{cases} \mathbf{y}'' + \mathbf{A}(\mathbf{y}, \boldsymbol{\zeta}) = \mathbf{0} & \text{in } Q, \\ \boldsymbol{\zeta}'' + \mathbf{B}(\mathbf{y}, \boldsymbol{\zeta}) = \mathbf{0} & \text{in } Q, \\ \mathbf{y} = \hat{\mathbf{y}}, \quad \boldsymbol{\zeta} = \hat{\boldsymbol{\zeta}} & \text{on } \Sigma, \\ \mathbf{y}(0) = \mathbf{y}^0, \quad \mathbf{y}'(0) = \mathbf{y}^1, \quad \boldsymbol{\zeta}(0) = \boldsymbol{\zeta}^0, \quad \boldsymbol{\zeta}'(0) = \boldsymbol{\zeta}^1 \quad \text{in } \omega. \end{cases}$$
(6)

The question we address in this paper is the classical one in exact controllability⁵: Is this system exactly controllable, i.e., does there exist a finite time $T^0 > 0$, a boundary control $(\hat{y}, \hat{\zeta})$ applied on Σ or on a part of it, and in which functional spaces, such that for initial conditions $(y^0, y^1), (\zeta^0, \zeta^1)$ in appropriate functional spaces the system is at rest after time T^0 ; in other words: y(T) = y'(T) = 0, $\zeta(T) = \zeta'(T) = 0$ in ω , for all $T \ge T^0$.

2. Properties of the homogeneous evolution problem

In this section we establish the main properties of the homogeneous evolution problem associated to (6):

$$\begin{cases} \boldsymbol{u}'' + \boldsymbol{A}(\boldsymbol{u}, \boldsymbol{\theta}) = \boldsymbol{0} & \text{in } \boldsymbol{Q}, \\ \boldsymbol{\theta}'' + \boldsymbol{B}(\boldsymbol{u}, \boldsymbol{\theta}) = \boldsymbol{0} & \text{in } \boldsymbol{Q}, \\ \boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{\theta} = \boldsymbol{0} & \text{on } \boldsymbol{\Sigma}, \\ \boldsymbol{u}(0) = \boldsymbol{u}^{0}, \quad \boldsymbol{u}'(0) = \boldsymbol{u}^{1}, \quad \boldsymbol{\theta}(0) = \boldsymbol{\theta}^{0}, \quad \boldsymbol{\theta}'(0) = \boldsymbol{\theta}^{1} \quad \text{in } \boldsymbol{\omega}. \end{cases}$$
(7)

Theorem 1. We assume the boundary $\partial \omega$ to be as smooth as necessary and we denote $\mathcal{H} = H_0^1(\omega) \times L^2(\omega), \mathcal{D} = H^2(\omega) \cap H_0^1(\omega) \times H_0^1(\omega)$.

(i-1) **Strong solution.** For initial conditions $(\boldsymbol{u}^0, \boldsymbol{u}^1) \in \mathcal{D}$ and $(\boldsymbol{\theta}^0, \boldsymbol{\theta}^1) \in \mathcal{D}$ there exists a strong solution to (7) that satisfies: $(\boldsymbol{u}, \boldsymbol{u}') \in \boldsymbol{C}^0(0, T; \mathcal{D}), (\boldsymbol{\theta}, \boldsymbol{\theta}') \in \boldsymbol{C}^0(0, T; \mathcal{D}).$

(i-2) Weak solution. For initial conditions $(\boldsymbol{u}^0, \boldsymbol{u}^1) \in \mathcal{H}$ and $(\boldsymbol{\theta}^0, \boldsymbol{\theta}^1) \in \mathcal{H}$, there exists a weak solution $(\boldsymbol{u}, \boldsymbol{u}') \in \mathcal{C}^0(0, T; \mathcal{H}), (\boldsymbol{\theta}, \boldsymbol{\theta}') \in \mathcal{C}^0(0, T; \mathcal{H})$ that satisfies:

$$\int_{Q} \left(\boldsymbol{u}' \cdot \boldsymbol{v}' + \boldsymbol{\theta}' \cdot \boldsymbol{\psi}' - \mathcal{A}(\boldsymbol{u}, \boldsymbol{\theta}; \boldsymbol{v}, \boldsymbol{\psi}) \right) \sqrt{a} \, \mathrm{d}x \, \mathrm{d}t = \int_{\omega} \left(\boldsymbol{u}' \cdot \boldsymbol{v} + \boldsymbol{\theta}' \cdot \boldsymbol{\psi} \right) \sqrt{a} \, \mathrm{d}x |_{0}^{T}$$
(8)

for all $(\boldsymbol{v}, \boldsymbol{v}') \in \boldsymbol{C}(0, T; \boldsymbol{\mathcal{H}}), (\boldsymbol{\psi}, \boldsymbol{\psi}') \in \boldsymbol{C}^0(0, T; \boldsymbol{\mathcal{H}})$, with the notation $v(t)|_0^T = v(T) - v(0)$.

(ii) **Stored elastic energy.** We define, for all $t \ge 0$, the stored elastic energy as $E(t) = \frac{1}{2} \int_{\omega} (|\boldsymbol{u}'|^2 + |\boldsymbol{\theta}'|^2 + \mathcal{A}(\boldsymbol{u}, \boldsymbol{\theta}; \boldsymbol{u}, \boldsymbol{\theta})) dx$. Hence, when no volume or surface forces are applied, the energy remains constant along all trajectories:

$$E(t) = E^{0} = \frac{1}{2} \int_{\omega} \left(\left| \boldsymbol{u}^{1} \right|^{2} + \left| \boldsymbol{\theta}^{1} \right|^{2} + \mathcal{A} \left(\boldsymbol{u}^{0}, \boldsymbol{\theta}^{0}; \boldsymbol{u}^{0}, \boldsymbol{\theta}^{0} \right) \right) \sqrt{a} \, dx \quad \text{for all } t \ge 0.$$
(9)

³ We denote the partial derivative with respect to time by superscript ', and the derivative with respect to space variables by ∂_i , for example $\boldsymbol{u}'' = \frac{\partial^2 \boldsymbol{u}}{\partial x^2}$, $\partial_{\alpha} \theta_{\beta} = \frac{\partial \theta_{\beta}}{\partial x_{\alpha}}$, div $\boldsymbol{q} = \partial_{\alpha} q_{\alpha}$.

⁴ For a complete expression of all the operators see [6].

⁵ In the sequel we skip the dependance on t and x whenever it is not necessary.

(iii) Maupertuis' principle. The weak solution satisfies:

$$\int_{Q} \left(|\boldsymbol{u}'|^2 + |\boldsymbol{\theta}'|^2 - \mathcal{A}(\boldsymbol{u},\boldsymbol{\theta};\boldsymbol{u},\boldsymbol{\theta}) \right) \sqrt{a} \, \mathrm{d}x \, \mathrm{d}t = \int_{\omega} \left(\boldsymbol{u}'(T) \cdot \boldsymbol{u}(T) + \boldsymbol{\theta}'(T) \cdot \boldsymbol{\theta}(T) - \boldsymbol{u}^1 \cdot \boldsymbol{u}^0 - \boldsymbol{\theta}^1 \cdot \boldsymbol{\theta}^0 \right) \sqrt{a} \, \mathrm{d}x.$$

(vi) **Direct inequality.** We consider multipliers $q \in C^2(\overline{\omega})$. The weak solution satisfies the identity

$$\frac{1}{2} \int_{\partial \omega} (\boldsymbol{q} \cdot \boldsymbol{n}) \mathcal{A}(\boldsymbol{u}, \boldsymbol{\theta}; \boldsymbol{u}, \boldsymbol{\theta}) \sqrt{a} \, \mathrm{d}\omega = \frac{1}{2} \int_{Q} \operatorname{div} \boldsymbol{q} (|\boldsymbol{u}'|^2 + |\boldsymbol{\theta}'|^2 - \mathcal{A}(\boldsymbol{u}, \boldsymbol{\theta}; \boldsymbol{u}, \boldsymbol{\theta})) \sqrt{a} \, \mathrm{d}x \, \mathrm{d}t \\ + \int_{\omega} (\boldsymbol{u}' \cdot \boldsymbol{q}_{\alpha} \partial_{\alpha} \boldsymbol{u} + \boldsymbol{\theta}' \cdot \boldsymbol{q}_{\alpha} \partial_{\alpha} \boldsymbol{\theta}) \sqrt{a} \, \mathrm{d}x |_{0}^{T} + \int_{Q} \mathcal{G}(\boldsymbol{q}, \boldsymbol{u}, \boldsymbol{\theta}) \sqrt{a} \, \mathrm{d}x \, \mathrm{d}t.$$
(10)

The expression of \mathcal{G} is given in the proof. There exists a positive constant $C(\omega, \boldsymbol{q})$ such that the weak solution satisfies the boundary inequality $\frac{1}{2} \int_{\partial \omega} (\boldsymbol{q} \cdot \boldsymbol{n}) \mathcal{A}(\boldsymbol{u}, \boldsymbol{\theta}; \boldsymbol{u}, \boldsymbol{\theta}) \sqrt{a} \, d\omega \leq C(\omega, \boldsymbol{q}) E^0(1+T)$. As a consequence $(\gamma_{\alpha\beta}, \gamma_{\alpha3}, r_{\alpha\beta})$ have a trace in $L^2(\omega)$.

(v) **An identity.** We consider the multiplier $\mathbf{p} = x - x^0 \in \mathbf{C}^2(\overline{\omega})$, and get for all weak solution

$$\frac{1}{2} \int_{\Sigma} (\boldsymbol{p} \cdot \boldsymbol{n}) \mathcal{A}(\boldsymbol{u}, \boldsymbol{\theta}; \boldsymbol{u}, \boldsymbol{\theta}) \sqrt{a} \, \mathrm{d}\omega \, \mathrm{d}t = E^0 T + \int_{\omega} \left(\boldsymbol{u}' \cdot \left(p_{\alpha} \partial_a \boldsymbol{u} + \frac{\boldsymbol{u}}{2} \right) + \boldsymbol{\theta}' \cdot \left(p_{\alpha} \partial_a \boldsymbol{\theta} + \frac{\boldsymbol{\theta}}{2} \right) \right) \sqrt{a} \, \mathrm{d}x|_0^T \\ + \int_{\Omega} \mathcal{K}(\boldsymbol{p}, \boldsymbol{u}, \boldsymbol{\theta}) \sqrt{a} \, \mathrm{d}x \, \mathrm{d}t,$$
(11)

where the expression of the operator \mathcal{K} is given in the proof.

(vi) **Observability condition.** There exists two positive constants $\tilde{K}(\omega, \boldsymbol{s}, k)$, $\tilde{T}(\omega, \boldsymbol{s}, k)$ which depends upon the domain ω , the immersion s and the shear correction coupling constant k, such that:

$$\frac{1}{2} \int_{\Sigma} (\boldsymbol{p} \cdot \boldsymbol{n}) \mathcal{A}(\boldsymbol{u}, \boldsymbol{\theta}; \boldsymbol{u}, \boldsymbol{\theta}) \sqrt{a} \, \mathrm{d}\omega \, \mathrm{d}t \ge \left(\left(1 - \tilde{K}(\omega, \boldsymbol{s}, k) \right) \left(T - \tilde{T}(\omega, \boldsymbol{s}, k) \right) \right) E^{0}.$$
(12)

2.1. Technical results

We establish some useful properties of the change of metric and change of curvature tensors γ and ρ . Let $\boldsymbol{q} \in \boldsymbol{C}^2(\omega)$ and $\boldsymbol{v} \in \boldsymbol{H}^1(\omega)$ then

$$\begin{cases} \gamma_{\alpha\beta}(q_{\alpha}\partial_{\alpha}\boldsymbol{v}) = q_{\tau}\partial_{\tau}\gamma_{\alpha\beta}(\boldsymbol{v}) + g_{\alpha\beta}(\boldsymbol{q},\boldsymbol{v}), \\ \gamma_{\alpha3}(q_{\alpha}\partial_{\alpha}\boldsymbol{v}, q_{\alpha}\partial_{\alpha}\boldsymbol{\psi}) = q_{\tau}\partial_{\tau}\gamma_{\alpha3}(\boldsymbol{v}, \boldsymbol{\psi}) + g_{\alpha3}(\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{\psi}) \\ \rho_{\alpha\beta}(q_{\alpha}\partial_{\alpha}\boldsymbol{v}, q_{\alpha}\partial_{\alpha}\boldsymbol{\psi}) = q_{\tau}\partial_{\tau}r_{\alpha\beta}(\boldsymbol{v}, \boldsymbol{\psi}) + r_{\alpha\beta}(\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{\psi}). \end{cases}$$

with

$$\begin{aligned} g_{\alpha\beta}(\boldsymbol{q},\boldsymbol{v}) &= 1/2(\partial_{\alpha}q_{\tau}\partial_{\tau}v_{\beta} + \partial_{\beta}q_{\tau}\partial_{\tau}v_{\alpha}) + v_{i}q_{\tau}\partial_{\tau}\Gamma_{\alpha\beta}^{t} \quad \text{with } \Gamma_{\alpha\beta}^{3} = b_{\alpha\beta}, \\ g_{\alpha3}(\boldsymbol{q},\boldsymbol{v},\boldsymbol{\psi}) &= 1/2(\partial_{\alpha}q_{\tau}\partial_{\tau}v_{3} - v_{\sigma}q_{\tau}\partial_{\tau}b_{\alpha}^{\sigma}), \\ r_{\alpha\beta}(\boldsymbol{q},\boldsymbol{v},\boldsymbol{\psi}) &= 1/2\partial_{\tau}v_{\sigma}\left(b_{\alpha}^{\sigma}\partial_{\beta}q_{\tau} + b_{\beta}^{\sigma}\partial_{\alpha}q_{\tau}\right) - 1/2q_{\tau}\left(\partial_{\tau}b_{\alpha}^{\sigma}\partial_{\beta}v_{\sigma} + \partial_{\tau}b_{\beta}^{\sigma}\partial_{\alpha}v_{\sigma}\right) - q_{\tau}v_{i}\partial_{\tau}R_{\alpha\beta}^{i} \\ &- 1/2(\partial_{\alpha}q_{\tau}\partial_{\tau}\psi_{\beta} + \partial_{\beta}q_{\tau}\partial_{\tau}\psi_{\alpha}) - \psi_{\sigma}q_{\tau}\partial_{\tau}\Gamma_{\alpha\beta}^{\sigma} \\ &\text{with } R_{\alpha\beta}^{\lambda} &= -1/2(b_{\alpha}^{\sigma}\Gamma_{\beta\sigma}^{\lambda} + b_{\beta}^{\sigma}\Gamma_{\alpha\sigma}^{\lambda}), R_{\alpha\beta}^{3} &= -b_{\alpha}^{\sigma}b_{\beta\sigma}. \end{aligned}$$

Let $\boldsymbol{p} = x - x^0$, then we get

$$f_{\alpha\beta}(\boldsymbol{p}, \boldsymbol{v}) = \gamma_{\alpha\beta}(\boldsymbol{v}) + h_{\alpha\beta}(\boldsymbol{p}, \boldsymbol{v}),$$

$$g_{\alpha3}(\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{\psi}) = \gamma_{\alpha3}(\boldsymbol{v}, \boldsymbol{\psi}) + h_{\alpha3}(\boldsymbol{p}, \boldsymbol{v}, \boldsymbol{\psi}),$$

$$\rho_{\alpha\beta}(\boldsymbol{p}, \boldsymbol{v}, \boldsymbol{\psi}) = \rho_{\alpha\beta}(\boldsymbol{v}, \boldsymbol{\psi}) + l_{\alpha\beta}(\boldsymbol{p}, \boldsymbol{v}, \boldsymbol{\psi}),$$

with

$$\begin{cases} h_{\alpha\beta}(\boldsymbol{p},\boldsymbol{v}) = v_i \big((1/2) p_\tau \partial_\tau \Gamma^l_{\alpha\beta} + \Gamma^l_{\alpha\beta} \big), \\ h_{\alpha3}(\boldsymbol{p},\boldsymbol{v},\boldsymbol{\psi}) = -1/2 \big(\psi_\alpha + v_\sigma (p_\tau \partial_\tau b^\sigma_\alpha + b^\sigma_\alpha) \big), \\ l_{\alpha\beta}(\boldsymbol{p},\boldsymbol{v},\boldsymbol{\psi}) = -R^i_{\alpha\beta} v_i - 1/2 p_\tau \big(\partial_\tau b^\sigma_\alpha \partial_\beta v_\sigma + \partial_\tau b^\sigma_\beta \partial_\alpha v_\sigma + 2 v_i \partial_\tau R^i_{\alpha\beta} + \psi_\sigma \partial_\tau \Gamma^\sigma_{\alpha\beta} \big) - \Gamma^\sigma_{\alpha\beta} \psi_\sigma. \end{cases}$$

2.2. Sketch of the proof of theorem

(i) Strong and weak solution. This is a direct application of Hille-Yoshida theory.

(ii) Stored elastic energy is obtained by multiplying the first equation of (7) by u', the second one by θ' and by integration. The final result is obtained by density.

(iii) Maupertuis' principle is obtained by multiplying the first equation of (7) by \boldsymbol{u} , the second one by $\boldsymbol{\theta}$ and by using the homogeneous boundary condition $\boldsymbol{u} = \boldsymbol{0}, \boldsymbol{\theta} = \boldsymbol{0}$ on $\partial \omega$. The final result is obtained by density.

(iv) **Direct inequality.** For the sake of clarity the proof is broken into 4 steps. First, we recall the identity $\partial_{\alpha}\sqrt{a} = \Gamma^{\beta}_{\alpha\beta}\sqrt{a}$. Step 1. For $\mathbf{q} \in \mathbf{C}^{2}(\omega)$ and \mathbf{u} a strong solution of (7), we have

$$\int_{Q} \boldsymbol{u}'' \cdot q_{\alpha} \partial_{\alpha} \boldsymbol{u} \sqrt{a} \, \mathrm{d}x \, \mathrm{d}t = \frac{1}{2} \int_{Q} \left(\partial_{\alpha} q_{\alpha} + q_{\alpha} \Gamma_{\alpha\beta}^{\beta} \right) \left| \boldsymbol{u}' \right|^{2} \sqrt{a} \, \mathrm{d}x \, \mathrm{d}t - \frac{1}{2} \int_{\Sigma} \left(\boldsymbol{q} \cdot \boldsymbol{n} \right) \left| \boldsymbol{u}' \right|^{2} \sqrt{a} \, \mathrm{d}\omega \, \mathrm{d}t + \int_{\omega} \boldsymbol{u}' \cdot q_{\alpha} \partial_{\alpha} \boldsymbol{u} \sqrt{a} \, \mathrm{d}x \right|_{0}^{T},$$
$$= \frac{1}{2} \int_{Q} \left(\operatorname{div} \boldsymbol{q} + q_{\alpha} \Gamma_{\alpha\beta}^{\beta} \right) \left| \boldsymbol{u}' \right|^{2} \sqrt{a} \, \mathrm{d}x \, \mathrm{d}t + \int_{\omega} \boldsymbol{u}' \cdot q_{\alpha} \partial_{\alpha} \boldsymbol{u} \sqrt{a} \, \mathrm{d}x \right|_{0}^{T},$$

since $\mathbf{u} = \mathbf{0}$ on $\partial \omega$ implies $\mathbf{u}' = \mathbf{0}$ on $\partial \omega$. By the same kind of computation, we obtain

$$\int_{Q} (\boldsymbol{u}'' \cdot q_{\alpha} \partial_{\alpha} \boldsymbol{u} + \boldsymbol{\theta}'' \cdot q_{\alpha} \partial_{\alpha} \boldsymbol{\theta}) \sqrt{a} \, \mathrm{d}x \, \mathrm{d}t = \frac{1}{2} \int_{Q} (\operatorname{div} \boldsymbol{q} + q_{\alpha} \Gamma_{\alpha\beta}^{\beta}) (|\boldsymbol{u}'|^{2} + |\boldsymbol{\theta}'|^{2}) \sqrt{a} \, \mathrm{d}x \, \mathrm{d}t \\ + \int_{\omega} (\boldsymbol{u}' \cdot q_{\alpha} \partial_{\alpha} \boldsymbol{u} + \boldsymbol{\theta}' \cdot q_{\alpha} \partial_{\alpha} \boldsymbol{\theta}) \sqrt{a} \, \mathrm{d}x|_{0}^{T}.$$
(13)

Step 2. Since $\mathbf{u} = \mathbf{0}$ on $\partial \omega$ implies $\partial_{\alpha} \mathbf{u} = n_{\alpha} \partial_{n} \mathbf{u}$ and $n_{\sigma} \partial_{\alpha} \mathbf{u} = n_{\alpha} \partial_{\sigma} \mathbf{u}$ on $\partial \omega$, we get by a straightforward computation

$$\int_{\omega} \left(\boldsymbol{A}(\boldsymbol{u},\boldsymbol{\theta}) \cdot \boldsymbol{q}_{\alpha} \partial_{\alpha} \boldsymbol{u} + \boldsymbol{B}(\boldsymbol{u},\boldsymbol{\theta}) \cdot \boldsymbol{q}_{\alpha} \partial_{\alpha} \boldsymbol{\theta} \right) \sqrt{a} \, \mathrm{d}\boldsymbol{x} = \int_{\omega} \mathcal{A}(\boldsymbol{u},\boldsymbol{\theta};\boldsymbol{q}_{\alpha} \partial_{\alpha} \boldsymbol{u},\boldsymbol{q}_{\alpha} \partial_{\alpha} \boldsymbol{\theta}) \sqrt{a} \, \mathrm{d}\boldsymbol{x} - \int_{\partial \omega} (\boldsymbol{q} \cdot \boldsymbol{n}) \, \mathcal{A}(\boldsymbol{u},\boldsymbol{\theta};\boldsymbol{u},\boldsymbol{\theta}) \sqrt{a} \, \mathrm{d}\boldsymbol{\omega}$$

Step 3. For all $\boldsymbol{q} \in \boldsymbol{C}^{2}(\omega)$ and $(\boldsymbol{v}, \boldsymbol{\psi}) \in \boldsymbol{H}^{1}(\omega) \times \boldsymbol{H}^{1}(\omega)$ we have

$$\int_{\omega} \mathcal{A}(\mathbf{v}, \boldsymbol{\psi}; q_{\alpha} \partial_{\alpha} \mathbf{v}, q_{\alpha} \partial_{\alpha} \boldsymbol{\psi}) \sqrt{a} \, \mathrm{d}x = \frac{1}{2} \int_{\partial \omega} (\mathbf{q} \cdot \mathbf{n}) \mathcal{A}(\mathbf{v}, \boldsymbol{\psi}; \mathbf{v}, \boldsymbol{\psi}) \sqrt{a} \, \mathrm{d}\omega - \frac{1}{2} \int_{\omega} \mathrm{div} \, \mathbf{q} \mathcal{A}(\mathbf{v}, \boldsymbol{\psi}; \mathbf{v}, \boldsymbol{\psi}) \sqrt{a} \, \mathrm{d}x + \int_{\omega} G(\mathbf{q}, \mathbf{v}, \boldsymbol{\psi}) \sqrt{a} \, \mathrm{d}x,$$

with

$$\begin{split} G(\boldsymbol{q},\boldsymbol{v},\boldsymbol{\psi}) &= \left(\boldsymbol{\varepsilon} a^{\alpha\beta\sigma\tau} \gamma_{\alpha\beta}(\boldsymbol{v}) g_{\sigma\tau}(\boldsymbol{q},\boldsymbol{v}) + \boldsymbol{\varepsilon} k a^{\alpha\beta3} \gamma_{\alpha3}(\boldsymbol{v},\boldsymbol{\psi}) g_{\beta3}(\boldsymbol{q},\boldsymbol{v},\boldsymbol{\psi}) + \frac{\boldsymbol{\varepsilon}^3}{3} a^{\alpha\beta\sigma\tau} \rho_{\alpha\beta}(\boldsymbol{v},\boldsymbol{\psi}) r_{\sigma\tau}(\boldsymbol{q},\boldsymbol{v},\boldsymbol{\psi})\right) \\ &- \frac{1}{2} q_{\lambda} \left(\boldsymbol{\varepsilon} \partial_{\lambda} \left(a^{\alpha\beta\sigma\tau} \sqrt{a}\right) \gamma_{\alpha\beta}(\boldsymbol{v}) \gamma_{\sigma\tau}(\boldsymbol{v}) + \boldsymbol{\varepsilon} k \partial_{\lambda} \left(a^{\alpha\beta3} \sqrt{a}\right) \gamma_{\alpha3}(\boldsymbol{v},\boldsymbol{\psi}) \gamma_{\beta3}(\boldsymbol{v},\boldsymbol{\psi}) \\ &+ \frac{\boldsymbol{\varepsilon}^3}{3} \partial_{\lambda} \left(a^{\alpha\beta\sigma\tau} \sqrt{a}\right) \rho_{\alpha\beta}(\boldsymbol{v},\boldsymbol{\psi}) \rho_{\sigma\tau}(\boldsymbol{v},\boldsymbol{\psi}) \right). \end{split}$$

The direct equality is obtained with $\mathcal{G}(\boldsymbol{q}, \boldsymbol{u}, \boldsymbol{\theta}) = G(\boldsymbol{q}, \boldsymbol{u}, \boldsymbol{\theta}) + \frac{1}{2} q_{\alpha} \Gamma_{\alpha\beta}^{\beta} (|\boldsymbol{u}'|^2 + |\boldsymbol{\theta}'|^2).$

Step 4. The majoration is a consequence of Korn's inequality.

(v) An identity. We replace q by $p = x - x^0$ in the previous expression of G and get $G(p, v, \psi) = \mathcal{A}(v, \psi; v, \psi) + K(p, v, \psi)$, with

$$\begin{split} K(\boldsymbol{p},\boldsymbol{v},\boldsymbol{\psi}) &= \left(\boldsymbol{\varepsilon} a^{\alpha\beta\sigma\tau} \gamma_{\alpha\beta}(\boldsymbol{v}) h_{\sigma\tau}(\boldsymbol{p},\boldsymbol{v}) + \boldsymbol{\varepsilon} k a^{\alpha\beta3} \gamma_{\alpha3}(\boldsymbol{v},\boldsymbol{\psi}) h_{\beta3}(\boldsymbol{p},\boldsymbol{v},\boldsymbol{\psi}) + \frac{\boldsymbol{\varepsilon}^3}{3} a^{\alpha\beta\sigma\tau} \rho_{\alpha\beta}(\boldsymbol{v},\boldsymbol{\psi}) l_{\sigma\tau}(\boldsymbol{p},\boldsymbol{v},\boldsymbol{\psi})\right) \\ &- \frac{1}{2} p_{\lambda} \bigg(\boldsymbol{\varepsilon} \partial_{\lambda} \big(a^{\alpha\beta\sigma\tau} \sqrt{a} \big) \gamma_{\alpha\beta}(\boldsymbol{v}) \gamma_{\sigma\tau}(\boldsymbol{v}) + \boldsymbol{\varepsilon} k \partial_{\lambda} \big(a^{\alpha\beta3} \sqrt{a} \big) \gamma_{\alpha3}(\boldsymbol{v},\boldsymbol{\psi}) \gamma_{\beta3}(\boldsymbol{v},\boldsymbol{\psi}) \\ &+ \frac{\boldsymbol{\varepsilon}^3}{3} \partial_{\lambda} \big(a^{\alpha\beta\sigma\tau} \sqrt{a} \big) \rho_{\alpha\beta}(\boldsymbol{v},\boldsymbol{\psi}) \rho_{\sigma\tau}(\boldsymbol{v},\boldsymbol{\psi}) \bigg). \end{split}$$

Next, by recalling that $\partial_{\alpha} p_{\beta} = \delta^{\beta}_{\alpha}$, and $\partial_{\alpha} p_{\alpha} = 2$ in dimension 2, and by adapting [4], we obtain the result with $\mathcal{K}(\boldsymbol{p}, \boldsymbol{u}, \boldsymbol{\theta}) = K(\boldsymbol{p}, \boldsymbol{u}, \boldsymbol{\theta}) + \frac{1}{2} p_{\alpha} \Gamma^{\beta}_{\alpha\beta} (|\boldsymbol{u}'|^2 + |\boldsymbol{\theta}'|^2)$.

(vi) Observability condition. Following [4] we establish the existence of a positive constant $\tilde{C}(\omega, \mathbf{s}, k)$ such that:

$$\left| \int_{\omega} \left(\mathbf{u}' \cdot \left(p_{\alpha} \partial_{\alpha} \mathbf{u} + \frac{\mathbf{u}}{2} \right) + \theta' \cdot \left(p_{\alpha} \partial_{\alpha} \theta + \frac{\theta}{2} \right) \right) \sqrt{a} \, \mathrm{d}x \right|$$

$$\leq \frac{C(\omega, \mathbf{s}, k)}{2} \int_{\omega} \left(\left| \mathbf{u}' \right|^2 + \left| \theta' \right|^2 \right) \sqrt{a} \, \mathrm{d}x + \frac{R^2}{2C(\omega, \mathbf{s}, k)} \int_{\omega} \mathcal{A}(\mathbf{u}, \theta; \mathbf{u}, \theta) \sqrt{a} \, \mathrm{d}x \leq \tilde{C}(\omega, \mathbf{s}, k) E^0,$$

where $R = \max_{x \in \overline{\omega}} |x - x_0|$ and $C(\omega, \mathbf{s}, k) > 0$ is related to Korn's constant which depends upon ω .

For a mapping $s \in C^4(\overline{\omega})$ there exists a positive bound $\tilde{K}(\omega, \mathbf{s}, k)$ such that

$$\int_{Q} \mathcal{K}(\boldsymbol{p}, \boldsymbol{u}, \boldsymbol{\theta}) \sqrt{a} \, \mathrm{d}x \leq \frac{\tilde{K}(\omega, \boldsymbol{s}, k)}{2}(\omega) \int_{Q} \left(\mathcal{A}(\boldsymbol{u}, \boldsymbol{\theta}; \boldsymbol{u}, \boldsymbol{\theta}) + \left(\left| \boldsymbol{u}' \right|^{2} + \left| \boldsymbol{\theta}' \right|^{2} \right) \right) \sqrt{a} \, \mathrm{d}x = T \tilde{K}(\omega, \boldsymbol{s}, k) E^{0},$$

and the proof is completed with $\tilde{T}(\omega, \mathbf{s}, k) = \frac{2\tilde{C}(\omega, \mathbf{s}, k)}{1 - \tilde{K}(\omega, \mathbf{s}, k)}$.

3. Exact controllability

We consider the partition of the boundary $\partial \omega = \partial \omega^+ \cup \partial \omega^-$ be such that $\mathbf{p} \cdot \mathbf{n} > \mathbf{0}$ on $\partial \omega^+$ and $\mathbf{p} \cdot \mathbf{n} < \mathbf{0}$ on $\partial \omega^-$. Let $\mathcal{H}' = \mathbf{L}^2(\omega) \times \mathbf{H}^{-1}(\omega)$.

Theorem 2. We assume that the initial conditions for problem (6) are $(\mathbf{y}^0, \mathbf{y}^1) \in \mathcal{H}', (\boldsymbol{\zeta}^0, \boldsymbol{\zeta}^1) \in \mathcal{H}'$. When $\tilde{K}(\omega, \mathbf{s}, k) < 1$, there exists a minimum time $T^0 = T^0(\omega, \mathbf{s}) > 0$ and a control $(\hat{\mathbf{y}}, \hat{\mathbf{\zeta}}) \in \mathbf{L}^2(\Sigma^+) \times \mathbf{L}^2(\Sigma^+)$

 $\hat{\boldsymbol{y}} = \partial_{\boldsymbol{n}} \boldsymbol{u}, \qquad \hat{\boldsymbol{\zeta}} = \partial_{\boldsymbol{n}} \boldsymbol{\theta} \quad on \ \boldsymbol{\Sigma}^+ = \partial \omega^+ \times (0, T),$

where $(\mathbf{u}, \boldsymbol{\theta})$ is the unique solution to the homogeneous evolution problem (7) associated to initial conditions $(\mathbf{u}^0, \mathbf{u}^1) \in \mathcal{H}$, $(\boldsymbol{\theta}^0, \boldsymbol{\theta}^1) \in \mathcal{H}$. This control drives system (6) to rest for $T \ge T^0$.

Proof. This is a consequence of the observability condition. In addition, by using the results of [4] we show that the condition $\tilde{K}(\omega, \mathbf{s}, k) < 1$ means that the shell is not "too far" from a plane and that the coupling factor k is "small". By extending the results of [2,3], for Reissner-Mindlin plates, we show that we can get rid of the condition on k (see [6] for the complete computation of this condition for a cylindrical shell).

Let us remark that stabilization for this kind of coupled system can be studied as in [5].

Acknowledgements

Part of this research was done while the first author was visiting the LNCC/MCT (Brasil) in April 2009 with partial support of the program PCI. The second author was partially supported by a grant of the Brazilian Government, CNPq, Proc. 305086/2006-5.

References

[1] M. Bernadou, P.G. Ciarlet, B. Miara, Existence theorems for two-dimensional linear shell theory, J. Elasticity 34 (1994) 111-138.

[2] V. Komornik, Contrôlabilité exacte en temps minimal de quelques modèles de plaques, C. R. Acad. Sci. Paris, Ser. I 307 (1988) 471-474.

- [3] V. Komornik, A new method of exact controllability in short time and applications, Ann. Fac. Sci. Toulouse Math. 10 (3) (1989) 415-464.
- [4] B. Miara, V. Valente, Exact controllability of a Koiter Shell by a boundary action, J. Elasticity 52 (1999) 267-287.
- [5] B. Miara, M.L. Santos, Energy decay in piezoelectric systems, Applicable Analysis (2010), submitted for publication.
- [6] B. Miara, G. Perla-Menzala, Exact controllability of Naghdi shells. Theory and Numerical Simulation, work in preparation.