Partial Differential Equations/Mathematical Physics

The Goursat problem for the Einstein–Yang–Mills–Higgs system in weighted Sobolev spaces

Problème de Goursat pour les équations d'Einstein–Yang–Mills–Higgs dans les espaces de Sobolev à poids

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We establish original Moser estimates to clarify and complete previous works of Christodoulou and Müller zum Hagen concerning local existence and uniqueness results for the Goursat problem associated to second order quasilinear hyperbolic systems. As an application we locally solve, in some weighted Sobolev spaces, the Goursat problem for the Einstein–Yang–Mills–Higgs system using harmonic and Lorentz gauges.

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1. Spaces of functions used and Moser estimates

\( L \) denotes a compact domain of \( \mathbb{R}^{n+1} \) \((n \geq 2)\), with a piecewise smooth boundary \( \partial L \). For \( \omega = 1, 2 \), define \( G^\omega = \{ x \in L : x^\omega = 0 \} \), where \( x = (x^i)_{i=1,\ldots,n+1} \) is the global canonical coordinates system of \( \mathbb{R}^{n+1} \). Assume that \( G^1 \cup G^2 \subset \partial L \). Define a time-function \( v(x) \) of \( \tau(x) = x^1 + x^2 \). For \( s \in \mathbb{N} \) and \( t \in (0, T_0) \) where \( T_0 = \sup x < L \tau(x) \), define the following point sets and weighted norms as in [7]:

\[
L_t = \{ x \in L : 0 \leq \tau(x) \leq t \}, \quad \mathcal{A}_t = \{ x \in L : \tau(x) = t \}, \quad G^\omega_t = \{ x \in G^\omega : 0 \leq \tau(x) \leq t \},
\]

\[
\| v \|_{H^1(S_t, R)} = \| v \|_{H^1(S^t)} = \left( \sum_{k=0}^{s} \int_{S_t} |D^k_R v|^2 \ dx \right)^{\frac{1}{2}},
\]

\[
\| v \|_{E^1(S_t)} = \| v \|_{E^1(S^t)} = \sup_{0 \leq \sigma \leq t} \| v \|_{\Sigma^t, S_t}, \quad \text{with } S_t = \bigcup_{0 \leq \sigma \leq t} \Sigma^t,
\]

where \( \alpha = \frac{1}{2} \) if \( S_t \in \{ \mathcal{A}_t, G^\omega_t \}, \alpha = 1 \) if \( S_t = L_t, \alpha = 0 \) if \( S_t = \mathcal{A}_t \). \( R \) is a surface (submanifold) of \( \mathbb{R}^{n+1} \) such that \( S_t \subset R \subset L \). \( D^k_R v \) are \( k \)-th order derivatives (in distributional sense) tangent to \( R \). \( D^k_R v \) is the norm of \( D^k_R v \) w.r.t. the Kronecker metric \( \delta^{ab} \). \( D_t = \frac{\partial}{\partial t} \), \( D_{St} \) is the volume element induced on \( S_t \) by \( dx_1 \ldots dx^{n+1} \).

We also define further norms as seen [7]:

\[
\| v \|_{E^1(G^\omega_t)} = \| v \|_{E^1(G^\omega_t)} = \sum_{k=0}^{s} \left( \int_{S_t} |D^k_R v|^{2(s-k)} \ dx \right)^{\frac{1}{2}}, \quad \| v \|_{E^1(L_t)} = \left( \int_{S_t} |v|^{2} \ dx \right)^{\frac{1}{2}}, \quad \| v \|_{E^1(L)} = \left( \int_{S_t} |v|^{2} \ dx \right)^{\frac{1}{2}},
\]

\[
\| v \|_{E^1(G^\omega_t)} = \left( \int_{S_t} |v|^{2} \ dx \right)^{\frac{1}{2}}, \quad \| v \|_{E^1(G^\omega_t)} = \left( \int_{S_t} |v|^{2} \ dx \right)^{\frac{1}{2}}, \quad \text{such that } \| v \|_{E^1(L_t)} = \sup_{0 \leq \sigma \leq t} \| v \|_{\Sigma^t, S_t}.
\]

\( C^\infty(L_t) \) denotes the space of restrictions to \( L_t \) of functions which are \( C^\infty \) on \( \mathbb{R}^{n+1} \). \( H_1(L_t) \) and \( H_2(L_t) \) are Hilbert spaces and \( C^\infty(L_t) \) is dense in \( H_1(L_t) \). \( E_1(L_t) \) and \( E_2(L_t) \) are Banach spaces. We will also need the following Banach spaces \( K_1(G^\omega_1) = \{ v \in E_1(G^\omega_1) : D_2 v \in E_1(G^\omega_1) \}, \]

\( K_2(G^\omega_2) = \{ v \in E_1(G^\omega_2) : D_1 v \in E_1(G^\omega_2) \} \) with their respective norms \( \| v \|_{K_i(G^\omega_i)} = \| v \|_{L^2}, \| v \|_{K_i(G^\omega_i)} = \| v \|_{L^2} \). Let \( W \) be an open subset of \( \mathbb{R}^l \), \( l \in \mathbb{N}^* \), with \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} \). \( C_0^\infty(L_t \times W) \) denotes the space of functions \( f : L_t \times W \to \mathbb{R} \) such that \( D^i f \) exists (in the usual sense) for all \( i = 0, \ldots, k \) and are continuous and bounded on \( L_t \times W \). \( C_0^\infty(L_t \times W) \) is endowed with the norm \( \| f \|_{C_0^\infty(L_t \times W)} = \sup_{(x,w) \in L_t \times W} |x + f|_{C_0^\infty(L_t \times W)} \).

The following Moser estimates play a crucial role in the resolution of the quasilinear Goursat problem (initial data are assigned on two intersecting smooth null hypersurfaces). The detailed proofs are provided in [9] by adapting the tools of [4,5].

**Theorem 1.1.** Let \( u_i \in E_1(L_t) \), \( i = 1, \ldots, l \); \( W \) an open subset of \( \mathbb{R}^l \): \( f : L_t \times W \to \mathbb{R} \) such that \( f \in C^2_{0} \) \((L_t \times W)\); \( u : L_t \to W \) a mapping \( u(x) = (u_1(x), \ldots, u_l(x)) \). We assume that \( 1 \leq m \leq s, n < 2s \). Then the function \( f(x, u(x)) \) of \( x \) satisfies the following inequality

\[
\left\| \frac{f(x, u(x))}{L^2_{1,m}} \right\|_{L^2_{1,m}} \leq c_t(m, s) \left( \| f \|_{C^2_{0} \times W} \left[ 1 + \| u \|_{L^2_{1,s}} \right] \right)^{2m-1},
\]

\( c_t(m, s) \) being a non-decreasing function of \( t \), depending also on \( m \) and \( s \).

**Theorem 1.2.** (i) Let \( u_i, v_i \in E_1(L_t) \), \( i = 1, \ldots, l \); \( u, v : L_t \to \mathbb{R} \) two mappings, \( u(x) = (u_1(x), \ldots, u_l(x)) \) and \( v(x) = (v_1(x), \ldots, v_l(x)) \): \( f : L_t \times W \to \mathbb{R} \) such that \( f \in C^2_{0} \) \((L_t \times W)\), \( f(x, 0) = 0 \) and \( D_{0} f \in C_{b}^{2m-3} \times W \), \( m = 1, \ldots, l \); \( W \) an open subset of \( \mathbb{R}^l \) such that \( u(x) + \theta(v(x) - u(x)) \in W, \forall \theta \in [0, 1] \). We assume that \( 2 \leq m \leq s \); \( n + 1 < 2(s - 1) \). Then the following inequality is satisfied
\[ \| f(x, v(x)) - f(x, u(x)) \|_{L_{t}, m-1} \leq c_{t}(m, s) \max_{1 \leq i \leq l} \| D_{u} f \|_{C^{2m-3}([L_{t}, W])} \left[ 1 + \| u \|_{L_{t}, s} + \| v \|_{L_{t}, s} \right]^{2m-3} \| v - u \|_{L_{t}, s-1}, \] (2)

c_{t}(m, s) being a non-decreasing function of \( t \), depending also on \( m \) and \( s \).

(ii) If in addition to the assumptions in (i) we assume that \( m \leq s - 1 \), \( f \in C^{2m-3}_{b}(L_{t} \times W) \), \( D_{u} f \in C^{2m-1}_{b}(L_{t} \times W) \), \( V_{i} = 1, \ldots, l \), and \( w \in E_{s-2}(L_{t}) \), then it holds that
\[ \| f(x, v(x)) - f(x, u(x)) \|_{L_{t}, m-1} \leq c_{t}(m, s) \max_{1 \leq i \leq l} \| D_{u} f \|_{C^{2m-1}([L_{t}, W])} \left[ 1 + \| u \|_{L_{t}, s} + \| v \|_{L_{t}, s} \right]^{2m-1} \| v - u \|_{L_{t}, s-1} \| w \|_{L_{t}, s-2}. \] (3)

2. The quasilinear Goursat problem: statement and proof of the result

The following quasilinear Cauchy problem is considered with unknown \( u \)
\[ g^{ab}(x, u) D_{ab} u = f(x, u, Du) \quad \text{in } L_{T}, \quad u = u_{\omega} \quad \text{on } G^{\omega}_{T}, \] (4)
where \( T \in (0, T_{0}] \), \( u = (u^{A}), Du = (D_{a}u^{A}) = (\frac{\partial u^{A}}{\partial x^{a}}) \), \( D_{ab} u = (\frac{\partial^{2} u^{A}}{\partial x^{a} \partial x^{b}}) \), \( f = (f^{i}), \quad u = (u^{j}), \quad \omega = 1, 2, a, b, \ldots = 1, \ldots, n + 1, \quad A, I, J, \ldots = 1, \ldots, N \). Einstein summation convention is understood.

**Assumptions** \((q_{s})\), \( \frac{n}{2} + 2 < s \in \mathbb{N} \)

- \( g^{ab}(x, u) \in C^{2s-2}(U \times V) \), where \( U \) is an open domain of \( \mathbb{R}^{n+1} \) and \( V \) is an open domain of \( \mathbb{R}^{N} \), such that \( L \subset U \), \( 0 \in V \) and \( g^{ab}(x, 0) = \gamma^{ab} \) with \( \gamma^{ab} \) defined in (6).
- \( g^{ab}(x, u) \) is regularly hyperbolic (see [7]) with hyperbolic constant \( h \) independent of \((x, u)\).
- \( f(x, u, Du) \in C^{2s-4}(U \times V \times W) \), \( f(x, 0, 0) = 0 \), where \( W \) is an open domain of \( \mathbb{R}^{N(n+1)} \).
- \( u \) is continuous on \( G^{\omega}_{T} \) and \( G^{\omega}_{T} \) is characteristic w.r.t. \( g^{ab}(x, u(x)), u(G^{\omega}_{T}) \subset V \), \( u \in E_{2s-1}(G_{T}^{\omega}), [u:r] \in H_{2s-1}(\Gamma), \quad u = u_{\omega} \) on \( \Gamma \).

**Theorem 2.1.** (i) Under assumptions \((q_{s})\), there exists \( T_{2} \in (0, T] \) such that the quasilinear Goursat problem (4) has in \( L_{T_{2}} \) a unique solution \( u \in \mathbb{E}_{s}(L_{T_{2}}) \).

(ii) There exists a positive real number \( d \) such that if \( \sum_{j=1}^{2} | u_{\omega} |_{C^{2s}_{T}} < d \), then the solution in (i) is global, i.e. \( T_{2} = T \).

**Proof.** The proof of item (i) is sketched in four main steps. The details are provided in [9]. Define \( \mathbb{E}_{s}(L_{T}) = \{ w \in \mathbb{E}_{s}(L_{T}) \} \) and consider the following mapping \( \kappa : \mathbb{E}_{s}(L_{T}) \rightarrow \mathbb{E}_{s}(L_{T}) \), where \( u \) solves the following linear Goursat problem
\[ g^{ab}(x, w) D_{ab} w = f(x, w, Dw) \quad \text{in } L_{T}, \quad u = u_{\omega} \quad \text{on } G^{\omega}_{T}. \] (5)

At the first step, thanks to Theorem 8.1 of [7], we construct an element \( w_{1} \) of \( \mathbb{E}_{s}(L_{T}) \) as the solution of the following linear Goursat problem
\[ \gamma^{ab} D_{ab} w = 0 \quad \text{in } L_{T}, \quad w = u_{\omega} \quad \text{on } G^{\omega}_{T}, \] (6)
where \( \gamma^{ab} = -1 \) if \( (a, b) = (1, 2) \) or \( (2, 1) \), \( \gamma^{ab} = 1 \) if \( a = b = 3, \ldots, n + 1 \), \( \gamma^{ab} = 0 \) elsewhere. At the second step, by using Sobolev inequalities in Section 3 of [7] and Moser estimate (1) of Theorem 1.1, we show that the mapping \( \kappa \) is well defined. At the third step we prove that \( \kappa \) is a contraction from a ball of \( \mathbb{E}_{s}(L_{T_{2}}) \) into itself. To do this, take \( w \in \mathbb{E}_{s}(L_{T}), u = \kappa(w), \) then use once more Sobolev inequalities and Theorem 8.1 of [7] and Moser estimate (1) of Theorem 1.1 to get the following inequality for \( t \in (0, T) \):
\[ \| u \|_{L_{t}, s} \leq C_{1} \left[ \sum_{j=1}^{2} \| u_{\omega} \|_{C^{2s}_{t} L_{T}, 2s-1} + t^{\frac{1}{2}} \| f \|_{C^{2s-3}([L_{t}, L_{T}], x \times W, B_{1}, B_{2}^{*}, \tau_{1}, \tau_{2}^{*})} \left[ 1 + \| w \|_{L_{t}, s} \right]^{2s-3} \right]. \] (7)
where the constant \( C_{1} > 0 \) does not depend on \( t \), \( B_{1} \) and \( B_{2}^{*} \) are such that \( B_{1} \times B_{2}^{*} \subset V \times W \) where \( V \) and \( W \) are defined in assumptions \((q_{s})\). Taking \( R = \max(\| w_{1} \|_{L_{t}, s}, 2C_{1} \sum_{j=1}^{2} \| u_{\omega} \|_{C^{2s}_{T}, 2s-1}), \) it follows from (7) that there exists \( T_{1} \leq T \) such that
\[ \| w \|_{L_{t}, s} \leq R \quad \Rightarrow \quad \| u \|_{L_{t}, s} \leq R \quad \text{for } t \leq T_{1}. \] (8)
Now take \( w_{1}, w_{2} \in \mathbb{E}_{s}(L_{T}) \) such that \( \| w_{1} \|_{L_{t}, s} \leq R \) and \( u_{1} = \kappa(w_{1}), i = 1, 2 \). By using Sobolev inequalities of [7] and Moser estimates (2) and (3) of Theorem 1.2, we gain the following inequality for \( 0 < t \leq T_{1} \):
where the constant $C_1 > 0$ does not depend on $t$. It follows from (9) that there exists a positive real number $T_2 \leq T_1$ such that
\[
C T_2^\frac{1}{2} (1 + R) [1 + 2R]^{2s-5} \leq \frac{1}{2}.
\]

Set $B_{R,T_2} = \{ w \in \tilde{E}_{s}^T (L_2^s) : \| w \|_{L_T^2} \leq R \}$. $B_{R,T_2}$ is endowed with the distance defined by the norm $\| . \|_{L_T^2}$ to be a non-empty and complete metric space (thanks to weak compactness arguments as in [5, 7]). It follows from (8), (9) and (10) that $\kappa$ is a contraction from $B_{R,T_2}$ into itself. Thus $\kappa$ has a unique fixed point $u \in B_{R,T_2}$. $u$ is therefore a solution to the Goursat problem (4) in $E_s(L_2^s)$. At the fourth step the uniqueness of the solution of (4) in $E_s(L_2^s)$ is shown by using the energy inequality for the linear Goursat problem and Gronwall lemma as in [5,7]. The proof of item (ii) follows from item (i) through a judicious exploitation of hypothesis $f(x,0) = 0$ and by applying Taylor formula to the function $f(x,\ldots)$ in the neighborhood of $(u,Du) = (0,0) \in \mathbb{R}^N \times \mathbb{R}^{(n+1)N}$. $\square$

3. Application to the Einstein–Yang–Mills–Higgs system (EYMH)

Here we discuss the local resolution of the Goursat problem for the EYMH system. Throughout all the section Roman indices vary from 1 to 4 whereas Greek indices vary from 3 to 4. Denote by $(\xi')$ the local coordinates in an unknown 4-dimensional manifold $M$ endowed with an unknown Lorentzian metric $g$. Let $(\varepsilon_i)_{i=1 \ldots N}$ be an orthogonal basis of an $\mathbb{R}$-valued functions defined on $M$ with values in $\mathbb{R}$. $A$ is locally defined by $A = \Lambda_i^j dx^j \otimes \varepsilon_i$, where $\Lambda_i^j$ are $\mathbb{R}$-valued functions defined on $M$. Consider the unknown Yang–Mills field $F$ (the curvature of $A$) which is locally defined by $F_{ij} = \nabla_i A_j - \nabla_j A_i + [A_i, A_j]$, where $\nabla$ denotes the covariant derivative w.r.t. the space–time metric $g$. Consider an unknown Higgs field which is an unknown $G$-valued function $\Phi$ defined on $M$. The EYMH system reads as follows

\[
R_{ij} - \frac{1}{2} R g_{ij} = \rho_{ij} \quad \text{(Einstein system)},
\]
\[
\nabla_i F^i_j = f^j \quad \text{(Yang–Mills system)},
\]
\[
\nabla_i \nabla^i \Phi = H \quad \text{(Higgs system)},
\]

where $(R_{ij})$ and $R$ are respectively the Ricci tensor and the scalar curvature of the unknown metric $g$. $(\rho_{ij})$ is the energy–momentum tensor given by $\rho_{ij} = F_{ik} F^k_j - \frac{1}{4} g_{ij} F_{kl} F^{kl} + \Phi_{ij}$, where $\Phi_{ij} = \nabla_i \Phi \nabla^i \Phi - \frac{1}{2} g_{ij} (\nabla_k \Phi \nabla^k \Phi + V(\Phi^2))$, $\Phi^2 = \Phi \cdot \Phi$, $V$ is a given $C^\infty$ real valued function defined on $\mathbb{R}$. $(J^i)$ is the Yang–Mills current given by $J^i = [\Phi, \nabla^i \Phi]$. $\nabla_i$ is the gauge covariant derivative or the Yang–Mills operator, it acts on $\Phi$ and $F^i_j$ as follows: $\nabla_i \Phi = \nabla_i \Phi + [A_i, \Phi]$, $\nabla_i F^i_j = \nabla_i F^i_j + [A_i, F^i_j]$. $H$ is a known $C^\infty$ $G$-valued function given by (see [1]) $H^i = V'(\Phi^2) \delta^i$, where $V'$ is the derivative of $V$. Throughout the remainder of the paper comma ”,” denotes usual partial derivative (e.g. $\frac{\partial \Phi}{\partial x} = \Phi_i$). Assume that the following harmonic and Lorentz gauges conditions are fulfilled

\[
J^i_k \equiv g^{km} J^i_m = 0, \quad \Delta = \nabla_k \nabla^k = 0,
\]

where $\Gamma^i_m$ are the Christoffel symbols of the unknown metric $g$. Then the Einstein–Yang–Mills–Higgs system (11) reduces to the following hyperbolic quasilinear form with unknown $u = (g_{ij}, A_p, \Phi)$:

\[
g^{km} D_{km} u = f(u, Du),
\]

where the non-linearity $f(u, Du) = (f_{ij}(u, Du), f_p(u, Du), \Psi(u, Du))$ is given as in [9] by $f_{ij}(u, Du) = Q_{ij}(g, Dg) - 2 \rho_{ij} + 2 g_{ij}(g^{km} D_{km} u)$, where $Q_{ij}$ depends quadratically on $Dg$.

\[
f_p(u, Du) = f_p(A, \Phi, D\Phi) - (g^{ki}_{,p} A_{k,i} + g^{ki}[A_k, A_p, i])
\]
\[
+ g_{jp} [g^{ki} g^{ji}_{,k} A_l - A_k, l + [A_k, A_l]] + \Gamma^i_{jm} F^{jm} + \Gamma^i_{jm} F^{jm} + [A_i, F^i_j]],
\]
\[
\Psi(u, Du) = H(\Phi) - (2[A_i, \nabla^i \Phi] + [A_i, [A^i, \Phi]]).
\]

The following theorem summarizes the resolution of the Goursat problem for the EYMH system in weighted spaces defined in Section 1. Both the evolution and the constraints problems are involved. The improvement here is that the data are constructed for the EYMH model and are of finite differentiability order whereas those of [3,8] are $C^\infty$ and constructed either for the vacuum Einstein, Einstein–perfect fluid models or the EYM model.
Theorem 3.1. Let \( T \in (0, T_0) \) be a real number, \( p \geq 4 \) an integer and \( \omega \in \{1,2\} \). Let \( h^\omega, h^\omega, h^\omega \in K_{2p-1}(G_{T_1}^\omega) \) be given functions on \( G_{T_1}^\omega \) which constitute two symmetric positive definite matrices with determinant 1 at each point and \( (h^\omega, h^\omega, h^\omega) = (h^\omega, h^\omega, h^\omega) \) on \( \Gamma \). Let \( \Phi, \tilde{\Phi}, \tilde{\Phi}, \tilde{\Phi} \in K_{2p-1}(G_{T_1}^\omega) \) such that \( (\Phi, \tilde{\Phi}, \tilde{\Phi}, \tilde{\Phi}) \) on \( \Gamma \). Let \( \tilde{T}, \tilde{T}, \tilde{T}, \tilde{T} \in H_{2p-1}(\Gamma) \). Then there exists \( T_1 \in (0, T) \); \( \Phi, \tilde{\Phi}, \tilde{\Phi}, \tilde{\Phi} \in E_{2p-1}(G_{T_1}^\omega) \); \( \Phi, \tilde{\Phi}, \tilde{\Phi}, \tilde{\Phi} \in E_{2p-1}(G_{T_1}^\omega) \) such that

\[
\begin{align*}
(1) & \quad g^\omega_{ij} = \Omega^\omega h^\omega_{ij} \quad \text{on } G_{T_1}^\omega, \\
(2) & \quad u = (g^\omega, \tilde{\Phi}, \tilde{\Phi}, \tilde{\Phi}) \text{ is the solution of the complete EYMH system (11) with initial data } u = (g^\omega, \tilde{\Phi}, \tilde{\Phi}, \tilde{\Phi}) \text{ on } G_{T_1}^\omega, \\
(3) & \quad \Omega^\omega = \tilde{T} ; \quad \Omega^\omega = \tilde{T} ; \quad \Omega^\omega = \tilde{T} ; \quad g_{13,2} = \tilde{b} ; \quad g_{12,2} = \tilde{b} ; \quad A_{1,2} = \tilde{A} \quad \text{on } \Gamma.
\end{align*}
\]

Proof. We begin by discussing the resolution of the evolution problem, i.e. the resolution of the reduced EYMH system (13). Setting \( k_{ij} = g_{ij} - \gamma_{ij} \) where \( \gamma_{ij} \) is the inverse of \( \gamma^{ij} \) defined in (6), one obtains a convenient second order hyperbolic quasilinear system with unknown \( (k_{ij}, \tilde{A}, \tilde{A}) \) to which Theorem 2.1 applies. After the resolution of (13) we proceed to the resolution of the constraints problem. We just sketch how this problem is solved in three main steps. The details are provided in [9]. Firstly the hierarchical method of A.D. Rendall [8] is used to construct, from appropriate free data, all the remaining \( C^\infty \) initial data satisfying the gauge conditions \( F^k = 0 \) and \( \Delta = 0 \) on \( G \cup G^2 \). Secondly energy inequalities are established in appropriate weighted Sobolev spaces. Thirdly the \( C^\infty \) result and energy inequalities are combined to derive, thanks to a contraction argument, the construction of initial data in the weighted Sobolev spaces \( E_{2p-1}(G_{T_1}^\omega) \) that appear in the resolution of (13). Finally, since the gauge conditions are fulfilled, it turns out that the solution of (13) is also the solution of the complete EYMH system (11). □

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