Probability Theory

Uniqueness result for the BSDE whose generator is monotonic in $y$ and uniformly continuous in $z$

Un résultat d’unicité pour une équation différentielle stochastique rétrograde dont le générateur $g$ est monotone en $y$ et uniformément continue en $z$

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abstract

In this Note, we prove that if $g$ is continuous, monotonic and has a general growth in $y$, $g$ is uniformly continuous in $z$, and $(g(t,0,0))_{t\in[0,T]}$ is square integrable, then for each square integrable terminal condition $\xi$, the one-dimensional backward stochastic differential equation (BSDE) with the generator $g$ has a unique solution. This generalizes some corresponding (one-dimensional) results.

Résumé

Dans cette Note on démontre que si $g$ est continue, monotone, de croissance quelconque en $y$, $g$ uniformément continue en $z$ et $(g(t,0,0))_{t\in[0,T]}$ est de carré intégrable, alors pour toute condition finale $\xi$ de carré intégrable, en dimension un, l’équation différentielle stochastique rétrograde (BSDE) de générateur $g$, a une solution unique. Ce résultat généralise des résultats connus dans le cas de la dimension un.

1. Introduction

We consider the following one-dimensional backward stochastic differential equation (BSDE for short):

$$y_t = \xi + \int_t^T g(s, y_s, z_s) \, ds - \int_t^T z_s \cdot dB_s, \quad t \in [0, T].$$

where $\xi$ is a square integral random variable termed the terminal condition, the random function $g(\omega, t, y, z) : \Omega \times [0, T) \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is progressively measurable for each $(y, z)$, termed the generator of the BSDE (1), and $B$ is a $d$-dimensional Brownian motion. The solution $(y, z)$ is a pair of square integrable, adapted processes. The triple $(\xi, T, g)$ is called the parameters of the BSDE (1).
Such equations, in nonlinear case, were firstly introduced by [11], who established an existence and uniqueness result of a solution of the BSDE (1) under the Lipschitz assumption of the generator $g$. Since then, many efforts have been done in relaxing the Lipschitz hypothesis on $g$; see, for instance [10,9,8,1–3], etc. In particular, under the conditions that $g$ is continuous, monotonic and has a general growth in $y$, $g$ is Lipschitz continuous in $z$, and $(g(t, 0, 0))_{t \in [0, T]}$ is square integrable, [12] proved the existence and uniqueness of the solution to the BSDE (1). Furthermore, [4] proved the existence of the solution to the BSDE (1) if the above Lipschitz continuity condition is replaced with the continuity and linear growth condition. Recently, under the conditions that $g$ does not depend on $y$, $g$ is uniformly continuous in $z$, and $(g(t, 0))_{t \in [0, T]}$ is a bounded process, [7] obtained a uniqueness result on the solution of the BSDE (1).

Enlightened by these results, this Note proves that if $g$ is continuous, monotonic and has a general growth in $y$, $g$ is uniformly continuous in $z$, and the process $(g(t, 0, 0))_{t \in [0, T]}$ is square integrable, then for each square integrable terminal condition $\xi$, the BSDE (1) has a unique solution, which generalizes the corresponding (one-dimensional) results in [11,12,7]. It is worth mentioning that we use a different method from that used in [7], and our result does not need the condition that $(g(t, 0, 0))_{t \in [0, T]}$ is a bounded process.

2. Main result

Let $(\Omega, \mathcal{F}, P)$ be a probability space carrying a standard $d$-dimensional Brownian motion $(B_t)_{t \geq 0}$. Fix a terminal time $T > 0$, let $(\mathcal{F}_t)_{t \geq 0}$ be the natural $\sigma$-algebra generated by $(B_t)_{t \geq 0}$ and assume $\mathcal{F}_T = \mathcal{F}$. For every positive integer $n$, we use $|\cdot|$ to denote norm of Euclidean space $\mathbb{R}^n$. For $t \in [0, T]$, let $L^2(\Omega, \mathcal{F}_t, P)$ denote the set of all $\mathcal{F}_t$-measurable random variable $\xi$ such that $E|\xi|^2 < +\infty$. Let $L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$ denote the set of $\mathcal{F}_t$-progressively measurable, $\mathbb{R}^n$-valued process $\{X_t, t \in [0, T]\}$ such that

$$\|X\|_2 \triangleq \left( E \int_0^T |X_t|^2 \, dt \right)^{1/2} < +\infty.$$

Now, let $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ be a terminal condition, $g$ be the $\mathcal{F}_t$-progressively measurable generator of the BSDE (1). A solution of the BSDE (1) is a pair of processes $(y, z)$ in $L^2_{\mathcal{F}}(0, T; \mathbb{R}^{1+d})$ which satisfies BSDE (1) and $y$ is a continuous process. In this Note, we further assume that $g$ satisfies some of the following assumptions:

(H1) The process $(g(t, 0, 0))_{t \in [0, T]} \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^1)$.

(H2) $dP \times dt$-a.s., $(y, z) \mapsto f(\omega, t, y, z)$ is continuous.

(H3) $g$ is monotonic in $y$, i.e., there exists a constant $\mu \geq 0$, such that, $dP \times dt$-a.s.,

$$\forall y_1, y_2, z, \quad (g(\omega, t, y_1, z) - g(\omega, t, y_2, z))(y_1 - y_2) \leq \mu|y_1 - y_2|^2.$$

(H4) $g$ has a general growth with respect to $y$, i.e., $dP \times dt$-a.s.,

$$\forall y, \quad |g(\omega, t, y, 0)| \leq |g(\omega, t, 0, 0)| + \varphi(|y|),$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing continuous function.

(H5) $g$ is uniformly continuous in $z$ and uniform with respect to $(\omega, t, y)$, i.e., there exists a continuous, nondecreasing function $\phi(\cdot)$ from $\mathbb{R}_+$ to itself with at most linear growth and $\phi(0) = 0$ such that $dP \times dt$-a.s.,

$$\forall y, z_1, z_2, \quad |g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq \phi(|z_1 - z_2|).$$

Here and henceforth we denote the constant of linear growth for $\phi$ by $A$, i.e., $0 \leq \phi(x) \leq A(x + 1)$ for all $x \in \mathbb{R}_+$ (see [5] for details).

(H5') $g$ is Lipschitz continuous in $z$ and uniform with respect to $(\omega, t, y)$, i.e., there exists a constant $C \geq 0$ such that $dP \times dt$-a.s.,

$$\forall y, z_1, z_2, \quad |g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq C|z_1 - z_2|.$$

Remark 1. Under the conditions of (H1)–(H4) and (H5'), [12] established the existence and uniqueness of the solution to the BSDE with the generator $g$. This Note aims at establishing the existence and uniqueness under the conditions of (H1)–(H5). Obviously, (H5') can imply (H5).

In the following, we will put forward and prove our main result that if $g$ is continuous, monotonic and has a general growth in $y$, $g$ is uniformly continuous in $z$, and the process $(g(t, 0, 0))_{t \in [0, T]}$ is square integrable, then the BSDE with the generator $g$ has a unique solution, which generalizes the corresponding (one-dimensional) results in [11,12,7]. Rigorously, we have:
Theorem 1. Assume that $g$ satisfies (H1)–(H5). Then for each $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, the BSDE with parameters $(\xi, T, g)$ has a unique solution $(y^\xi, z^\xi)$.

Proof. Existence: Since $g$ satisfies (H4) and (H5), then $dP \times dt - a.s.$, for each $(y, z) \in \mathbb{R}^{1+d}$ we have
\[
|g(\omega, t, y, z)| \leq |g(\omega, t, y, 0)| + |g(\omega, t, 0, 0)| \leq \phi(|y|) + \phi(|z|)
\]
Thus the existence of the solution to the BSDE with parameters $(\xi, T, g)$ follows from Theorem 4.1 in [4].

Uniqueness: Assume that $(y, z)$ and $(y', z')$ be two solutions to the BSDE with parameters $(\xi, T, g)$ in $L^2(\Omega, \mathcal{T}; \mathbb{R}^{1+d})$.

Let $\hat{y} = y - y'$, $\hat{z} = z - z'$ then we have
\[
\hat{y}_t = \int_t^T \left[ g(s, y_s, z_s) - g(s, y'_s, z'_s) \right] ds - \int_t^T \hat{z}_s \cdot dB_s, \quad t \in [0, T].
\]

Using the Tanaka–Meyer formula (see [6]), one gets that for each $t \in [0, T]$,
\[
\left| \hat{y}_t \right| = \int_t^T \left[ \frac{\hat{y}_s}{|\hat{y}_s|} \cdot \phi(|\hat{z}_s|) \right] ds - \left( l^0_T - l^0_t \right) - \int_t^T \frac{\hat{y}_s}{|\hat{y}_s|} \cdot \left( f(s, y_s, z_s) - f(s, y'_s, z'_s) \right) ds,
\]

where $l^0_t$ is the local time of $\hat{y}_t$ at 0 and
\[
V_t = - \int_0^t \left[ \frac{\mu}{|\hat{y}_s|} \cdot \phi(|\hat{z}_s|) \right] ds + \int_0^t \frac{\hat{y}_s}{|\hat{y}_s|} \cdot \left( f(s, y_s, z_s) - f(s, y'_s, z'_s) \right) ds - l^0_t.
\]

Thanks to (H3) and (H5), we know that
\[
\hat{y}_s \left[ g(s, y_s, z_s) - g(s, y'_s, z'_s) \right] = \hat{y}_s \left[ g(s, y_s, z_s) - g(s, y'_s, z'_s) \right] + \hat{y}_s \left[ g(s, y'_s, z'_s) - g(s, y'_s, z'_s) \right] 
\]
\[
\leq \mu |\hat{y}_s|^2 + |\hat{y}_s| \phi(|\hat{z}_s|).
\]

This inequality combining that $l^0_t$ is a continuous increasing process yields that $(V_t)_{t \in [0, T]}$ is a continuous decreasing process with $V_0 = 0$. Moreover, from (2) one also knows that
\[
V_T = \hat{y}_0 - \int_0^T \left[ \frac{\mu}{|\hat{y}_s|} \cdot \phi(|\hat{z}_s|) \right] ds \leq \int_0^T \left[ \frac{\mu}{|\hat{y}_s|} \cdot \phi(|\hat{z}_s|) \right] ds + \int_0^T |\hat{z}_s|^2 ds < +\infty.
\]

In the following, for each $n \geq 1$, from [11], one knows that the following BSDE has a unique solution $(Y^n, Z^n) \in L^2(\Omega, \mathcal{T}; \mathbb{R}^{1+d})$:
\[
Y^n_t = \int_t^T \left[ \mu Y^n_s + (n + 2A) Z^n_s + \phi \left( \frac{2A}{n + 2A} \right) \right] ds - \int_t^T Z^n_s \cdot dB_s, \quad t \in [0, T].
\]

Recalling that $\phi(\cdot)$ is a nondecreasing function from $\mathbb{R}_+$ to itself with at most linear growth, one can prove that for each $n \in \mathbb{N}$,
\[
\phi(x) \leq (n + 2A)x + \phi \left( \frac{2A}{n + 2A} \right).
\]
holds true for each \( x \in \mathbb{R}_+ \). In fact, if \( 0 \leq x < \frac{2A}{n+2A} \), the conclusion is obvious considering \( \phi(\cdot) \) is nondecreasing. And, if \( \frac{2A}{n+2A} < x < 1 \), we have \((n+2A)x > 2A = A + A > Ax + A \geq \phi(x)\). Finally, in the case of \( x \geq 1 \), we also have \((n+2A)x > 2Ax = Ax + Ax \geq Ax + A \geq \phi(x)\). Therefore, for each \( n \geq 1 \), from (5) we have

\[
g'(s) := \mu |\hat{y}_s| + 1_{\hat{y}_s \neq 0} \phi(|\hat{z}_s|) \leq \mu |\hat{y}_s| + 1_{\hat{y}_s \neq 0} (n+2A) |\hat{z}_s| + \phi \left( \frac{2A}{n+2A} \right)
\]

\[
= \mu (\hat{y}_s) + (n + 2A) \frac{|\hat{y}_s|}{|\hat{y}_s|} 1_{\hat{y}_s \neq 0} |\hat{z}_s| + \phi \left( \frac{2A}{n+2A} \right).
\]

(6)

Obviously, \( g'(\cdot) \in L^2_s(0,T; \mathbb{R}^1) \). Thus, considering inequalities (3) and (6), and the fact that \((0 - V_t)_{t \in [0,T]}\) is a continuous increasing process, by using Comparison Theorem 1.3 in [13] to compare the solution of the BSDE (2) with the one of the BSDE (4), we know that for each \( n \geq 1 \) and \( t \in [0,T], |\hat{y}_t| \leq Y^n_t, dP - a.s. \)

On the other hand, one may verify directly that for each \( t \in [0,T], \)

\[
Y^n_t = \frac{1}{\mu} \left( e^{\mu(T-t)} - 1 \right) \phi \left( \frac{2A}{n+2A} \right) \text{ and } Z^n_t = 0.
\]

Thus since \( \phi(\cdot) \) is a continuous function and \( \phi(0) = 0 \), we have \( |\hat{y}_t| \leq \lim_{n \to \infty} Y^n_t = 0, dP - a.s. \)

Hence, for each \( t \in [0,T] \) we have, \( Y_t = y_t, dP - a.s. \). That is to say, the solution to the BSDE with parameters \((\xi, T, g)\) is unique. The proof of Theorem 1 is complete. \( \square \)

**Remark 2.** From the proof of Theorem 1, one can see that we need only the monotonicity condition in \( y \) (see (H3)) and uniform continuity condition in \( z \) (see (H5)) to ensure the uniqueness of the solution of the BSDE:

From Theorem 1 one can easily obtain the following Corollaries which can be regarded as the extensions of the corresponding (one-dimensional) results in [7,11].

**Corollary 1.** Assume that \( g \) satisfies (H1) and (H5). Moreover, let \( g \) be independent of \( y \). Then for each \( \xi \in L^2(\Omega, \mathcal{F}_T, P) \), the BSDE with parameters \((\xi, T, g)\) has a unique solution.

**Corollary 2.** Assume that \( g \) satisfies (H1) and (H5). Moreover, let \( g \) be Lipschitz continuous in \( y \). Then for each \( \xi \in L^2(\Omega, \mathcal{F}_T, P) \), the BSDE with parameters \((\xi, T, g)\) has a unique solution.

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**References**


