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Mathematical Problems in Mechanics

Radiation condition and uniqueness for the outgoing elastic wave in a half-plane with free boundary

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Abstract

In this Note we deduce an explicit Sommerfeld-type radiation condition which is convenient to prove the uniqueness for the time-harmonic outgoing wave problem in an isotropic elastic half-plane with free boundary condition. The expression is obtained from a rigorous asymptotic analysis of the associated Green's function. The main difficulty is that the free boundary condition allows the propagation of a Rayleigh wave which cannot be neglected in the far field expansion. We also give the existence result for this problem. ***To cite this article: M. Durán et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Condition de radiation et unicité pour l'onde élastique sortante dans un demi-plane avec frontière libre. Dans cette Note, nous exhibons une condition de radiation explicite, du type Sommerfeld, qui nous permet de montrer (dans le domaine fréquenciel) l'unicité des solutions du problème d'onde élastique sortante dans un demi-plan avec frontière libre. Cette expression est obtenue par une analyse asymptotique rigoureuse de la fonction de Green's associée. La difficulté principale est que la condition de bord de frontière libre permet la propagation d'une onde de Rayleigh qui n'est pas négligeable dans le champ lointain. Nous donnons également un résultat d'existence pour ce problème. ***Pour citer cet article : M. Durán et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Dealing with scattering by half-space problems, it has been observed that surface waves can propagate for certain boundary conditions. It is the case of the Helmholtz equation in half-spaces with passive or non-absorbing boundary [3,4], or the case of elastic wave scattering in half-spaces with free boundary condition or with impedance boundary condition [2]. A surface wave has exponential decay towards the interior of the media and propagates guided by the surface. When the surface is unbounded, we cannot despise the contribution of that surface waves at infinity. Since

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the scattering problems needs a radiation condition to ensure uniqueness, we observe immediately that neither the usual Sommerfeld radiation condition, nor the well-known Kupradze radiation condition [6] are sufficient to describe outgoing surface wave behavior. In fact, surface waves propagate with their own velocity, which is different from the velocities of volume waves. The article [3] provides a new radiation condition, well adapt to prove the uniqueness results for acoustic scattering problems in half-spaces including surface wave behavior. The aim of this research is to extend these last results to scattering problems of elastic waves in isotropic half-spaces where surface (Rayleigh) waves are excited. Our attention will be focused in the problem of outgoing time-harmonic elastic waves in isotropic half-planes with free boundary condition.

2. Mathematical settings

Consider an isotropic elastic media filling the whole half-plane $\mathbb{R}^2_+ := \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 / x_2 > 0\}$. We are interested to find in this domain, a solution of the time-harmonic elastic wave equation when the source is given by a local normal stress excitation at the surface $\Gamma := \{\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2 / x_2 = 0\}$. Our problem will be to find a mechanical displacement field $\boldsymbol{u} = (u_1, u_2) : \mathbb{R}^2_+ \to \mathbb{C}^2$ satisfying (for $i = 1, 2$) the equations

$$\begin{cases} \rho\omega^2 u_i + \displaystyle\sum_{j=1}^{2} \frac{\partial \sigma_{ij}(\boldsymbol{u})}{\partial x_j} = 0 & \text{in } \mathbb{R}^2_+, \\ -\sigma_{i2}(\boldsymbol{u}) = f_i & \text{on } \Gamma. \end{cases} \tag{1}$$

The stress tensor σ is defined by $\sigma_{ij} := 2\mu\varepsilon_{ij} + \lambda\,\mathrm{tr}(\varepsilon)\delta_{ij}$, where ε is the strain tensor given by the constitutive relation $\varepsilon_{ij} = \frac{1}{2}(\partial_{x_j} u_i + \partial_{x_i} u_j)$. The Lamé coefficients μ and λ are positive constants related to the characteristics of the elastic material, and so is the density ρ. The source function $\boldsymbol{f} = (f_1, f_2)$ must have compact support in Γ for our early computations and ω will denote the angular frequency of the elastic wave.

Given any bidimensional vector $\boldsymbol{v} = (v_1, v_2)$, we define its orthogonal by $\boldsymbol{v}^\perp := (v_2, -v_1)$. In the same way, the differential operator $\nabla^\perp$ will be defined by $\nabla^\perp := (\partial_{x_2}, -\partial_{x_1})$.

3. The Green's function

Consider a punctual excitation at $\boldsymbol{y} = (y_2, y_2) \in \mathbb{R}^2_+$. We will work with a Green's function matrix G with the property that each of its column vectors is an outgoing solution of the elastic system (1) with Dirac's delta right-hand side and with homogeneous boundary condition. In vectorial notation:

$$\begin{cases} \rho\omega^2 \boldsymbol{g} + (\lambda + 2\mu)\nabla \operatorname{div} \boldsymbol{g} - \mu\nabla^\perp \operatorname{div} \boldsymbol{g}^\perp = -\delta(\boldsymbol{x} - \boldsymbol{y})\boldsymbol{e} & \text{in } \mathbb{R}^2_+, \\ \sigma_{i2}(\boldsymbol{g}) = 0 & \text{over } \Gamma \ (i = 1, 2). \end{cases} \tag{2}$$

The first column vector of G is a solution of (2) for $\boldsymbol{e} = (1, 0)$; the second column vector of G is a solution of (2) for $\boldsymbol{e} = (0, 1)$.

The classical procedure to obtain an expression for $\boldsymbol{g}$, is to apply in Eq. (2) an horizontal Fourier transform in the x_1-variable. Next we solve the remaining ordinary differential system in x_2 and finally we apply the Fourier inversion formula to that solution, i.e. we obtain an expression like

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\boldsymbol{g}}(\xi, x_2, y_2) e^{-i\xi(x_1 - y_1)}\, d\xi,$$

where $\hat{\boldsymbol{g}}$ denotes the Fourier transform of $\boldsymbol{g}$.

4. Asymptotic analysis

In this section $\hat{\boldsymbol{r}} = (\cos\varphi, \sin\varphi)$ will denote the unitary radial vector for $\varphi \in [0, \pi]$. The idea is to obtain an asymptotic relation between $\boldsymbol{g}$ and $\sigma(\boldsymbol{g})\hat{\boldsymbol{r}}$ when $r := |\boldsymbol{x} - \boldsymbol{y}| \to +\infty$. The key to analyse their asymptotic behaviors is to use the Kupradze's decomposition $\boldsymbol{g} = \boldsymbol{g}^p + \boldsymbol{g}^s$, where

$$\boldsymbol{g}^p(\mathbf{x}, \mathbf{y}) := \frac{1}{k_s^2 - k_p^2}\big(\Delta_{\mathbf{x}} + k_s^2\big)\boldsymbol{g}(\mathbf{x}, \mathbf{y}) \quad \text{and} \quad \boldsymbol{g}^s(\mathbf{x}, \mathbf{y}) := \frac{1}{k_p^2 - k_s^2}\big(\Delta_{\mathbf{x}} + k_p^2\big)\boldsymbol{g}(\mathbf{x}, \mathbf{y})$$

represent the pressure and shearing components respectively and k_p and k_s are the pressure and shearing wavenumbers. The next step is to study the asymptotic behavior of the integrals:

$$\boldsymbol{g}^j(\mathbf{x},\mathbf{y}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\boldsymbol{g}}^j(\xi, x_2, y_2) e^{-i\xi(x_1-y_1)}\,d\xi, \quad \text{where } j = p, s.$$

These integrals has the exponential term $e^{-\sqrt{\xi^2-k_j^2}x_2}$ as a factor, which induces different type of wave behavior depending if the square root is purely real or purely imaginary. Hence, it is convenient to split each projection $\boldsymbol{g}^j$ into:

$$\boldsymbol{g}^j_{\mathrm{v}}(\mathbf{x},\mathbf{y}) := \frac{1}{\sqrt{2\pi}} \int_{|\xi|<k_j} \hat{\boldsymbol{g}}^j e^{-i\xi(x_1-y_1)}\,d\xi \quad \text{and} \quad \boldsymbol{g}^j_{\mathrm{e}}(\mathbf{x},\mathbf{y}) := \frac{1}{\sqrt{2\pi}} \int_{|\xi|>k_j} \hat{\boldsymbol{g}}^j e^{-i\xi(x_1-y_1)}\,d\xi.$$

In that way, $\boldsymbol{g}_{\mathrm{v}} := \boldsymbol{g}^p_{\mathrm{v}} + \boldsymbol{g}^s_{\mathrm{v}}$ will be associated with (pressure and shearing) volume wave behavior and $\boldsymbol{g}_{\mathrm{e}} := \boldsymbol{g}^p_{\mathrm{e}} + \boldsymbol{g}^s_{\mathrm{e}}$ will be associated with evanescent and surface wave behavior.

The asymptotic contributions of $\boldsymbol{g}^j_{\mathrm{v}}$ and $\sigma(\boldsymbol{g}^j_{\mathrm{v}})\hat{\boldsymbol{r}}$ are obtained using the *stationary phase method* (cf. Evans [5]) and integration by parts arguments. They show outgoing wave behavior of the type

$$\sim \sin\varphi \frac{e^{ik_j r}}{\sqrt{r}} + O\big(r^{-1}\big). \tag{3}$$

Moreover, the main asymptotic contributions of $\boldsymbol{g}_{\mathrm{v}}$ and $\sigma(\boldsymbol{g}_{\mathrm{v}})\hat{\boldsymbol{r}}$ are multiplied by the factor $\sin\varphi$, which implies a decay as we approach the surface. As we hoped, $\boldsymbol{g}_{\mathrm{v}}$ and $\sigma(\boldsymbol{g}_{\mathrm{v}})\hat{\boldsymbol{r}}$ satisfy the following asymptotic relations:

$$\left(\sigma(\boldsymbol{g}_{\mathrm{v}})\hat{\boldsymbol{r}} - i\rho\frac{\omega^2}{k_p}\boldsymbol{g}_{\mathrm{v}}\right)\cdot\hat{\boldsymbol{r}} = O\big(r^{-1}\big) \quad \text{and} \quad \left(\sigma(\boldsymbol{g}_{\mathrm{v}})\hat{\boldsymbol{r}} - i\rho\frac{\omega^2}{k_s}\boldsymbol{g}_{\mathrm{v}}\right)\cdot\hat{\boldsymbol{r}}^{\perp} = O\big(r^{-1}\big), \quad \text{when } r \to +\infty. \tag{4}$$

These relations are similar to those in [1].

On the other hand, the main asymptotic terms of $\boldsymbol{g}_{\mathrm{e}}$ and $\sigma(\boldsymbol{g}_{\mathrm{e}})\hat{\boldsymbol{r}}$ are obtained using the *Cauchy's Residue Theorem*. The associated integrals have poles contributions at the wavenumber k_R of the Rayleigh surface wave (i.e. when $|\xi| = k_R > k_s > k_p$). We obtain outgoing surface wave behavior of the type

$$\sim e^{-\sqrt{k_R^2-k_j^2}x_2} e^{ik_R|x_1-y_1|} + O\big(r^{-1}\big). \tag{5}$$

The following asymptotic relation has been observed:

$$\sigma(\boldsymbol{g}_{\mathrm{e}})\boldsymbol{n} - iM\boldsymbol{g}_{\mathrm{e}} = O\big(r^{-1}\big), \quad \text{when } r \to +\infty, \tag{6}$$

where $\boldsymbol{n} = \operatorname{sgn}(x_1 - y_1)\mathbf{e}_1$ and M is the matrix

$$M = 2\mu k_R I + \frac{\mu\,\operatorname{sgn}(x_1-y_1)}{k_R^2 - \sqrt{k_R^2-k_p^2}\sqrt{k_R^2-k_s^2}} \begin{pmatrix} k_s^2 - 2k_p^2 & 0 \\ 0 & k_s^2 \end{pmatrix} \begin{pmatrix} \operatorname{sgn}(x_1-y_1)k_R & i\sqrt{k_R^2-k_s^2} \\ i\sqrt{k_R^2-k_p^2} & -\operatorname{sgn}(x_1-y_1)k_R \end{pmatrix}.$$

The matrix M satisfies the coercivity property $\Re\langle M\boldsymbol{u}, \bar{\boldsymbol{u}}\rangle \geqslant \beta\|\boldsymbol{u}\|^2$, for a positive constant $\beta > 0$.

5. The radiation condition

We conclude that the mean asymptotic terms of the volume waves and the surface wave contribute separately in two different regions of the half-plane. To explain that phenomena, we consider a parameter $\alpha \in (0, \frac{1}{2})$ and we define the sub-regions $\mathbb{R}^2_+(\alpha+) := \{(x_1, x_2) \in \mathbb{R}^2_+ / x_2 > r^\alpha\}$ and $\mathbb{R}^2_+(\alpha-) := \{(x_1, x_2) \in \mathbb{R}^2_+ / x_2 < r^\alpha\}$. Due to the exponential decay of the expression (5) when $x_2 \to +\infty$, we observe that in a region like $\mathbb{R}^2_+(\alpha+)$ the terms $\boldsymbol{g}_{\mathrm{e}}$ and $\sigma(\boldsymbol{g}_{\mathrm{e}})\hat{\boldsymbol{r}}$ behaves like $O(r^{-1})$ when $r \to +\infty$. On the other hand, in the complementary region $\mathbb{R}^2_+(\alpha-)$ the expression (3) tell us that $\boldsymbol{g}_{\mathrm{v}}$ and $\sigma(\boldsymbol{g}_{\mathrm{v}})\hat{\boldsymbol{r}}$ behave like $o(r^{-\frac{1}{2}})$ when $r \to +\infty$. Thus, adding the information of (4) and (6) we obtain that the radiation condition can be written as

$$\begin{cases} \left(\sigma(\mathbf{u})\hat{\boldsymbol{r}} - i\rho\frac{\omega^2}{k_p}\mathbf{u}\right)\cdot\hat{\boldsymbol{r}} = O\big(r^{-1}\big) & \text{in } \mathbb{R}^2_+(\alpha+), \\ \left(\sigma(\mathbf{u})\hat{\boldsymbol{r}} - i\rho\frac{\omega^2}{k_s}\mathbf{u}\right)\cdot\hat{\boldsymbol{r}}^{\perp} = O\big(r^{-1}\big) & \text{in } \mathbb{R}^2_+(\alpha+), \quad \text{when } r \to +\infty. \\ \big(\sigma(\mathbf{u})\hat{\boldsymbol{r}} - iM\mathbf{u}\big) = o\big(r^{-\frac{1}{2}}\big) & \text{in } \mathbb{R}^2_+(\alpha-), \end{cases} \tag{7}$$

Note that the expression (7) is well adapted for integration by part computations on the upper half-circle.

6. Functional spaces

Let $D'(\mathbb{R}^2_+)$ the space of distributions over $\mathbb{R}^2_+$. Consider the weighted Sobolev space:

$$W_\varrho := \left\{ v \in D'(\mathbb{R}^2_+)\colon\ \frac{v}{\sqrt{\varrho}\log\varrho} \in L^2(\mathbb{R}^2_+) \text{ and } \frac{\nabla v}{\sqrt{\varrho}\log\varrho} \in \left[L^2(\mathbb{R}^2_+)\right]^2 \right\},$$

where we have used the classic weight functions $\varrho := \sqrt{1+r^2}$ and $\log\varrho := \log(2+r^2)$. An accurate description of the space where we will look for a solution must consider the asymptotic behaviors studied in Sections 4 and 5. As we have seen, the worse behavior in terms of integrability at infinity, was the one given by the surface wave, which is characterized by expression (5). Nevertheless, this undesired behavior occurs only in the horizontal directions (in the vertical direction we have exponential decay). Behaviors like (5) are admitted in the space W_ϱ.

On the other hand, the volume wave behavior is characterized by expression (3) and it is also admitted in W_ϱ. Thus, if we consider now the radiation condition, a functional space which is adapted to our problem is:

$$W(\mathbb{R}^2_+) := \left\{ \mathbf{u} = (u_1, u_2)\colon\ \begin{array}{l} u_1, u_2, \operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{u}^\perp \in W_\varrho \\ \mathbf{u} \text{ satisfies the radiation condition} \end{array} \right\}.$$

7. The existence and uniqueness results

The existence is shown using the single layer potential

$$\boldsymbol{u}(\boldsymbol{x}) = \int_{\operatorname{supp} f} G(\boldsymbol{x}; y_1, 0)\,\boldsymbol{f}(y_1)\,dy_1,$$

which is in our space $W(\mathbb{R}^2_+)$ and satisfies the elastic system (1) thanks to the Green's function. The uniqueness result follows the same procedure used in [3]. First we prove that a radiating solution of the homogeneous problem (1) satisfies the estimations

$$\lim_{R\to+\infty} \|\boldsymbol{u}\|_{L^2(S_R^+)} = 0 \quad \text{and} \quad \lim_{R\to+\infty} \left\|\sigma(\boldsymbol{u})\hat{r}\right\|_{L^2(S_R^+)} = 0,$$

where S_R^+ represents the upper half-circumference of radius R. Then we show an important orthogonality property of the traces of our solution $\boldsymbol{u}$ with the following gradients of Bessel functions (in polar coordinates):

$$\left\{\nabla\left[J_n(k_p r)\cos(n\varphi)\right]\right\}_{n\geqslant 0} \quad \text{and} \quad \left\{\nabla\left[J_n(k_s r)\cos(n\varphi)\right]\right\}_{n\geqslant 0}.$$

This last information implies that the Fourier transform of the solution vanishes almost everywhere. The remaining spatial solution do not satisfy the radiation condition unless $u \equiv 0$.

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