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C. R. Acad. Sci. Paris, Ser. I 347 (2009) 1261–1264

COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

A non-existence result for the Ginzburg–Landau equations

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Received 15 July 2009; accepted after revision 23 September 2009

Available online 9 October 2009

Presented by Haïm Brezis

Abstract

We consider the stationary Ginzburg–Landau equations in $\mathbb{R}^d$, $d = 2, 3$. We exhibit a class of applied magnetic fields (including constant fields) such that the Ginzburg–Landau equations do not admit finite energy solutions. ***To cite this article: A. Kachmar, M. Persson, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Un résultat de non-existence pour les équations de Ginzburg–Landau. Nous considérons les équations de Ginzburg–Landau dans $\mathbb{R}^d$, $d = 2, 3$. Nous exhibons une classe de champs magnétiques appliqués telle que les équations de Ginzburg–Landau n'admettent pas de solution d'énergie finie. ***Pour citer cet article : A. Kachmar, M. Persson, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

The aim of the present note is to study the Ginzburg–Landau system of equations in $\mathbb{R}^2$,

$$\begin{cases} -(\nabla - iA)^2\psi = (1 - |\psi|^2)\psi, \\ -\nabla^\perp(\operatorname{curl} A - H) = \operatorname{Im}(\psi\overline{(\nabla - iA)\psi}). \end{cases} \tag{1}$$

Here $\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{C})$ is the complex order parameter, $A \in H^1_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ is the magnetic vector potential, $\operatorname{curl} A$ is the induced magnetic field

$$B = \operatorname{curl} A = \partial_{x_1} A_2 - \partial_{x_2} A_1, \tag{2}$$

$H \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ is the applied magnetic field, and $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ is the Hodge gradient.

Solutions of (1) are of particular interest in the physics literature as they do include periodic solutions with vortices distributed in a uniform lattice, named as Abrikosov's solution. We refer the reader to [1] for the physical motivation and to [2,4] for mathematical results in that direction.

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.09.024

Eqs. (1) are formally the Euler–Lagrange equations of the following Ginzburg–Landau energy,

$$\mathcal{G}(\psi, A) = \int\limits_{\mathbb{R}^2} \left(\left|(\nabla - iA)\psi\right|^2 + \frac{1}{2}\left(1 - |\psi|^2\right)^2 + |\operatorname{curl} A - H|^2 \right) \mathrm{d}x. \tag{3}$$

A solution (ψ, A) of (1) is said to have finite energy if $\mathcal{G}(\psi, A) < \infty$. When the applied magnetic field $H \in L^2(\mathbb{R}^2)$, it is proved in [6,8] that the system (1) admits finite energy solutions. In the present note, we would like to discuss the optimality of the hypothesis $H \in L^2(\mathbb{R}^2)$ thereby establishing negative results when this hypothesis is violated.

Our result is that if H is not allowed to decay fast at infinity (especially if it is constant), then there are no finite energy solutions to (1):

Theorem 1. *Let $\alpha < 1$. Assume that the applied magnetic field $H \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ and that there exist constants $R_0 > 0$ and $h > 0$ such that $H(x) \geqslant \frac{h}{|x|^\alpha}$ for all x with $|x| > R_0$. Then the Ginzburg–Landau system* (1) *does not admit finite energy solutions.*

Remark 2. We note that $\frac{1}{|x|^\alpha} \in L^2(\mathbb{R}^2 \setminus B(0, 1))$ if and only if $\alpha > 1$, which means that the result in Theorem 1 is really complementary to the results in [6,8].

Remark 3. The same non-existing result still holds if we instead impose the following properties on H: (1) $H \notin L^2(\mathbb{R}^2)$, (2) there exists $R_0 > 0$ such that for $H(x)$ is positive for $|x| > R_0$, and (3) there exists $R_1 > 0$ such that the reverse Hölder-inequality

$$\int\limits_{B(0,R)} H(x)\,\mathrm{d}x \geqslant \left|B(0, R)\right|^{1/2} \left(\int\limits_{B(0,R)} H(x)^2\,\mathrm{d}x \right)^{1/2} \tag{4}$$

holds for all $R > R_1$. The proof follows the proof of Theorem 1 until the end, where the alternative properties of H are used.

We conclude by mentioning an immediate generalization to the 3-dimensional equations. Let $\mathbf{H} = (H_1, H_2, H_3) \in L^2_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ be a given vector field. Consider the Ginzburg–Landau equations in $\mathbb{R}^3$,

$$\begin{cases} -(\nabla - iA)^2\psi = (1 - |\psi|^2)\psi, \\ -\operatorname{curl}(\operatorname{curl} A - H) = \operatorname{Im}(\psi\overline{(\nabla - iA)\psi}). \end{cases} \tag{5}$$

A solution $(\psi, A) \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{C}) \times H^1_{\mathrm{loc}}(\mathbb{R}^3; R^3)$ is said to have finite energy if $\mathcal{E}(\psi, A) = \int_{\mathbb{R}^3}(|(\nabla - iA)\psi|^2 + \frac{1}{2}(1 - |\psi|^2)^2 + |\operatorname{curl} A - H|^2)\,\mathrm{d}x < \infty$. We have then a similar result to Theorem 1.

Theorem 4. *Let $\alpha < \frac{3}{2}$. Assume that there exist $h > 0$ and $R_0 > 0$ such that the applied magnetic field $\mathbf{H} = (H_1, H_2, H_3) \in L^2_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3)$ satisfies, $H_3(x) \geqslant \frac{h}{|x|^\alpha} \forall x$ such that $|x| \geqslant R_0$. Then the Ginzburg–Landau system* (5) *does not admit finite energy solutions.*

Remark 5. Remark 3 carries over to three dimensions, but with any component H_j in place of H.

The proof of Theorem 4 is exactly the same as that of Theorem 1. So, we will give details only for the proof of Theorem 1. The essential key for proving Theorem 1 is a result from the spectral theory of magnetic Schrödinger operators stated in Lemma 7 below.

2. Two auxiliary lemmas

We start with the following observation concerning the Ginzburg–Landau system (1):

Lemma 6. *Assume that $H \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$. Let (ψ, A) be a weak solution of* (1) *such that $\mathcal{G}(\psi, A) < \infty$. Then $|\psi| \leqslant 1$ in $\mathbb{R}^2$.*

Proof. This result was proved by Yang [7, Lemma 3.1] for $\mathbb{R}^3$ under the assumption $H \in L^2(\mathbb{R}^3)$. The assumption on H is not used in Yang's proof but the proof only relies on the fact that the energy of (ψ, A) is finite. The proof of this lemma is line-by-line the same as [7], but with $\mathbb{R}^2$ in place of $\mathbb{R}^3$. $\square$

A key-ingredient is the following result from the spectral theory of magnetic Schrödinger operators.

Let χ be a cut-off function such that $0 \leqslant \chi \leqslant 1$, $\chi = 1$ in $[0, \frac{1}{2}]$ and $\chi = 0$ in $[1, \infty)$. For all $R > 0$, we introduce the function,

$$\chi_R(x) = \chi\left(\frac{|x|}{R}\right), \quad \forall x \in \mathbb{R}^2. \tag{6}$$

Lemma 7. *There exists a constant $C > 0$ such that, for all $\psi \in H^1(\mathbb{R}^2; \mathbb{C})$, $A \in H^1_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ and $R > 0$, the following inequality holds,*

$$\int_{B(0,R)} \left|(\nabla - iA)\psi\right|^2 \mathrm{d}x \geqslant \frac{1}{2} \int_{B(0,R)} B(x)|\chi_R\psi|^2 \mathrm{d}x - \frac{C}{R^2} \int_{B(0,R)\setminus B(0,R/2)} \left|\psi(x)\right|^2 \mathrm{d}x.$$

Here $B = \operatorname{curl} A$ and χ_R the function from (6).

Proof. We write,

$$\int_{B(0,R)} \left|(\nabla - iA)\psi\right|^2 \mathrm{d}x \geqslant \int_{B(0,R)} \left|\chi_R(\nabla - iA)\psi\right|^2 \mathrm{d}x \geqslant \frac{1}{2} \int_{B(0,R)} \left|(\nabla - iA)(\chi_R\psi)\right|^2 \mathrm{d}x - \int_{B(0,R)} |\psi\nabla\chi_R|^2 \mathrm{d}x.$$

To finish the proof, we just use the following well known inequality (see [3] or [5, Lemma 2.4.1]),

$$\int_{B(0,R)} \left|(\nabla - iA)\phi\right|^2 \mathrm{d}x \geqslant \pm \int_{B(0,R)} B(x)|\phi|^2 \mathrm{d}x, \quad \forall \phi \in H^1_0\big(B(0,R)\big). \qquad \square$$

3. Proof of Theorem 1

Assume that (ψ, A) is a finite energy solution of (1). Thanks to Lemma 6 we have $|\psi| \leqslant 1$ in $\mathbb{R}^2$.

Recalling the hypothesis on the applied magnetic field H that we assumed in Theorem 1, we may pick $R_0 > 0$ such that

$$H(x) \geqslant \frac{h}{|x|^\alpha}, \quad \forall |x| \geqslant R_0. \tag{7}$$

Applying Lemma 7, with (ψ, A) as above, a solution of (1), we obtain with $B = \operatorname{curl} A$,

$$\int_{\mathbb{R}^2} \left|(\nabla - iA)\psi\right|^2 \mathrm{d}x \geqslant \frac{1}{2} \int_{B(0,R)} B(x)|\chi_R\psi|^2 \mathrm{d}x - \frac{C}{R^2} \int_{B(0,R)\setminus B(0,R/2)} |\psi|^2 \mathrm{d}x.$$

Let $R > 2R_0$ and $\Omega_R = \{x \in \mathbb{R}^2\colon\ R_0 < |x| < R\}$. Then we may write,

$$\int_{\mathbb{R}^2} \left|(\nabla - iA)\psi\right|^2 \mathrm{d}x \geqslant \frac{1}{2} \int_{\Omega_R} B(x)|\chi_R\psi|^2 \mathrm{d}x + \frac{1}{2} \int_{B(0,R_0)} B(x)|\chi_R\psi|^2 \mathrm{d}x - \frac{C}{R^2} \int_{B(0,R)\setminus B(0,R/2)} |\psi|^2 \mathrm{d}x.$$

Using that $\int_{\mathbb{R}^2} |(\nabla - iA)\psi|^2 \mathrm{d}x \leqslant \mathcal{G}(\psi, A)$, $A \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $|\chi_R\psi| \leqslant 1$, we get a constant C_0 depending on R_0 such that,

$$\mathcal{G}(\psi, A) \geqslant \frac{1}{2} \int_{\Omega_R} B(x)|\chi_R\psi|^2 \mathrm{d}x - C_0. \tag{8}$$

So, let us handle the first term in the right-hand side above. We write,

$$\int_{\Omega_R} B(x)|\chi_R\psi|^2\,\mathrm{d}x = \int_{\Omega_R} H(x)|\chi_R\psi|^2\,\mathrm{d}x + \int_{\Omega_R} \big(B(x)-H(x)\big)|\chi_R\psi|^2\,\mathrm{d}x. \tag{9}$$

In order to handle the last term on the right of (9), we apply a Cauchy–Schwarz inequality and use the fact that $|\chi_R\psi| \leqslant 1$. In this way we get,

$$\left|\int_{\Omega_R} \big(B(x)-H(x)\big)|\chi_R\psi|^2\,\mathrm{d}x\right| \leqslant \left(\int_{\Omega_R} \big|B(x)-H(x)\big|^2\,\mathrm{d}x\right)^{1/2}\left(\int_{\Omega_R}\mathrm{d}x\right)^{1/2} \leqslant \big(\mathcal{G}(\psi,A)\big)^{1/2}|\Omega_R|^{1/2}.$$

Implementing this bound together with (7) in the right side of (8), we get the following lower bound,

$$\int_{\Omega_R} B(x)|\chi_R\psi|^2\,\mathrm{d}x \geqslant \int_{\Omega_R} \frac{h}{|x|^\alpha}|\chi_R\psi|^2\,\mathrm{d}x - \big(\mathcal{G}(\psi,A)\big)^{1/2}|\Omega_R|^{1/2}. \tag{10}$$

We need only to bound from below $\int_{\Omega_R}\frac{h}{|x|^\alpha}|\chi_R\psi|^2\,\mathrm{d}x$. Actually, using that $\chi_R = 1$ in $B(0,R/2)$ and a Cauchy–Schwarz inequality, we obtain,

$$\int_{\Omega_R} \frac{h}{|x|^\alpha}|\chi_R\psi|^2\,\mathrm{d}x \geqslant \int_{\Omega_{R/2}} \frac{h}{|x|^\alpha}|\psi|^2\,\mathrm{d}x = \int_{\Omega_{R/2}} \frac{h}{|x|^\alpha}\,\mathrm{d}x + \int_{\Omega_{R/2}} \frac{h}{|x|^\alpha}\big(|\psi|^2-1\big)\,\mathrm{d}x$$

$$\geqslant \frac{2\pi h}{2-\alpha}\big((R/2)^{2-\alpha} - R_0^{2-\alpha}\big) - \big(\mathcal{G}(\psi,A)\big)^{1/2} h\left(\frac{2\pi}{2-2\alpha}\right)^{1/2}\big((R/2)^{2-2\alpha} - R_0^{2-2\alpha}\big)^{1/2}.$$

Now we use the assumption that $\mathcal{G}(\psi,A) < \infty$. In this way, we get by implementing the right-hand side above in (10) and then by substituting the resulting lower bound into (8), a constant C such that,

$$\mathcal{G}(\psi,A) \geqslant \frac{2^{\alpha-1}\pi h}{2-\alpha}R^{2-\alpha} - CR^{1-\alpha} - CR - C. \tag{11}$$

Making $R\to\infty$ and recalling that $\alpha < 1$, we get a contradiction to the assumption that the energy $\mathcal{G}(\psi,A)$ is finite, thereby finishing the proof of Theorem 1.

Acknowledgements

The authors wish to thank the anonymous referee for valuable suggestions. A.K. is supported by a Starting Independent Researcher grant by the ERC under the FP7. M.P. is supported by the Lundbeck Foundation.

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