A minimum degree condition of fractional \((k, m)\)-deleted graphs

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Abstract

Let \(G\) be a graph of order \(n\), and let \(k \geq 1\) and \(m \geq 1\) be two integers. In this paper, we consider the relationship between the minimum degree \(\delta(G)\) and the fractional \((k, m)\)-deleted graphs. It is proved that if \(n \geq 4k - 5 + 2(2k + 1)m\) and \(\delta(G) \geq n/2\), then \(G\) is a fractional \((k, m)\)-deleted graph. Furthermore, we show that the minimum degree condition is sharp in some sense.

1. Introduction

The reader is referred to [1] for undefined terms and concepts. We consider finite undirected graphs without loops or multiple edges. Let \(G\) be a graph of order \(n\). We use \(V(G)\) and \(E(G)\) to denote its vertex set and edge set, respectively. For any \(x \in V(G)\), the degree of \(x\) in \(G\) is denoted by \(d_G(x)\). We write \(N_G(x)\) for the set of vertices adjacent to \(x\) in \(G\), and \(N_G[x]\) for \(N_G(x) \cup \{x\}\). For \(S \subseteq V(G)\), we write \(d_G(S)\) instead of \(\sum_{x \in S} d_G(x)\). We denote by \(G[S]\) the subgraph of \(G\) induced by \(S\), and \(G - S = G[V(G) \setminus S]\). Let \(S\) and \(T\) be two disjoint vertex subsets of \(G\), we use \(e_G(S, T)\) to denote the number of edges with one end in \(S\) and the other end in \(T\). If \(T = \{x\}\), then we write \(e_G(x, S)\) instead of \(e_G(T, S)\). We use \(\delta(G)\) for the minimum degree of \(G\).

Let \(k \geq 1\) be an integer. Then a spanning subgraph \(F\) of \(G\) is called a \(k\)-factor if \(d_F(x) = k\) for each \(x \in V(G)\). Let \(h : E(G) \rightarrow [0, 1]\) be a function. If \(\sum_{e \in x} h(e) = k\) holds for any \(x \in V(G)\), then we call \(G[F_h]\) a fractional \(k\)-factor

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of $G$ with indicator function $h$ where $F_h = \{ e \in E(G): h(e) > 0 \}$. In this paper we introduce firstly the definition of a fractional $(k, m)$-deleted graph, that is, a graph $G$ is called a fractional $(k, m)$-deleted graph if there exists a fractional $k$-factor $G(F_h)$ of $G$ with indicator function $h$ such that $h(e) = 0$ for any $e \in E(H)$, where $H$ is any subgraph of $G$ with $m$ edges. A fractional $(k, m)$-deleted graph is simply called a fractional $k$-deleted graph if $m = 1$.

Many authors have investigated $k$-factors or fractional $k$-factors [2,4–6]. The following results on $k$-factors and fractional $k$-factors are known:

**Theorem 1** (Katerinis [2]). Let $k \geq 1$ be an integer, and let $G$ be a graph of order $n$ with $n \geq 4k - 5$, kn even. If $\delta(G) \geq \frac{n}{2}$, then $G$ has a $k$-factor.

Yu showed a degree condition for the existence of a fractional $k$-factor.

**Theorem 2** (Yu [5]). Let $k$ be an integer with $k \geq 1$, and let $G$ be a connected graph of order $n$ with $n \geq 4k - 3$, $\delta(G) \geq k$. If $\max\{d_G(x), d_G(y)\} \geq \frac{k}{2}$ for each pair of nonadjacent vertices $x, y$ of $G$, then $G$ has a fractional $k$-factor.

From Theorem 2, we easily get the following result:

**Theorem 3.** Let $k \geq 1$ be an integer, and let $G$ be a connected graph of order $n$ with $n \geq 4k - 3$. If $\delta(G) \geq \frac{n}{2}$, then $G$ has a fractional $k$-factor.

The toughness $t(G)$ of a graph $G$ was defined as follows: $t(G) = \min\{\frac{|S|}{\omega(G-S)}: S \subseteq V(G), \omega(G-S) \geq 2\}$, if $G$ is not complete, where $\omega(G-S)$ denotes the number of components of $G-S$; otherwise, set $t(G) = +\infty$. Liu and Zhang gave a toughness condition for graphs to have fractional $k$-factors.

**Theorem 4** (Liu and Zhang [4]). Let $k \geq 2$ be an integer. A graph $G$ of order $n$ with $n \geq k + 1$ has a fractional $k$-factor if $t(G) \geq k - \frac{1}{k}$.

In this paper, we obtain a minimum degree condition for a graph to be a fractional $(k, m)$-deleted graph. Our result is an extension of Theorems 1 and 3.

**Theorem 5.** Let $k \geq 1$ and $m \geq 1$ be two integers. Let $G$ be a graph of order $n$ with $n \geq 4k - 5 + 2(2k + 1)m$. If $\delta(G) \geq \frac{n}{2}$, then $G$ is a fractional $(k, m)$-deleted graph.

In Theorem 5, if $m = 1$, then we get the following corollary:

**Corollary 1.** Let $k \geq 1$ be an integer. Let $G$ be a graph of order $n$ with $n \geq 8k - 3$. If $\delta(G) \geq \frac{n}{7}$, then $G$ is a fractional $k$-deleted graph.

2. The proof of Theorem 5

In order to prove Theorem 5, we depend on the following lemmas:

**Lemma 2.1** (Liu [3]). Let $G$ be a graph. Then $G$ has a fractional $k$-factor if and only if for every subset $S$ of $V(G)$, $\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \geq 0$, where $T = \{ x: x \in V(G) \setminus S, d_{G-S}(x) \leq k - 1 \}$.

**Lemma 2.2.** Let $k \geq 1$ and $m \geq 0$ be two integers, and let $G$ be a graph and $H$ a subgraph of $G$ with $m$ edges. Then $G$ is a fractional $(k, m)$-deleted graph if and only if for any subset $S$ of $V(G)$, $\delta_G(S, T) = k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| \geq \sum_{x \in T} d_H(x) - e_H(S, T)$, where $T = \{ x: x \in V(G) \setminus S, d_{G-S}(x) - d_H(x) + e_H(x, S) \leq k - 1 \}$.

**Proof.** Let $G' = G - E(H)$. Then $G$ is a fractional $(k, m)$-deleted graph if and only if $G'$ has a fractional $k$-factor. According to Lemma 2.1, this is true if and only if for any subset $S$ of $V(G)$, $\delta_G(S, T') = k|S| + d_{G'-S}(T') - k|T'| \geq 0$, where $T' = \{ x: x \in V(G) \setminus S, d_{G'-S}(x) \leq k - 1 \}$.
It is easy to see that \(d_{G-S}(x) = d_G(x) - d_H(x) + e_H(x, S)\) for any \(x \in T'\). By the definitions of \(T'\) and \(T\), we have \(T' = T\). Hence, we obtain \(\delta_G(S, T') = \delta_G(S, T) - \sum_{x \in T} d_H(x) + e_H(S, T)\). Thus, \(\delta_G(S, T') \geq 0\) if and only if \(\delta_G(S, T) \geq \sum_{x \in T} d_H(x) - e_H(S, T)\). It follows that \(G\) is a fractional \((k, m)\)-deleted graph if and only if \(\delta_G(S, T) = k|S| + \sum_{x \in T} d_{G-S}(x) - k|T| \geq \sum_{x \in T} d_H(x) - e_H(S, T)\). □

**Proof of Theorem 5.** Suppose that \(G\) satisfies the assumption of the theorem, but is not a fractional \((k, m)\)-deleted graph. Then by Lemma 2.2, there exists some subset \(S\) of \(V(G)\) such that

\[
k|S| + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - k) \leq -1,
\]

where \(T = \{x \in V(G) \setminus S \mid d_{G-S}(x) - d_H(x) + e_H(x, S) \leq k - 1\}\).

At first, we prove the following claims:

**Claim 1.** \(|S| \geq 1\).

**Proof.** If \(S = \emptyset\), then by (1), we have \(-1 \geq \sum_{x \in T} (d_{G-S}(x) - d_H(x) - k) \geq \sum_{x \in T} (\delta(G) - m - k) \geq 0\), this is a contradiction. □

**Claim 2.** \(|T| \geq k + 1\).

**Proof.** If \(|T| \leq k\), then by (1), Claim 1 and \(\delta(G) \geq \frac{n}{2}\), we have

\[
-1 \geq k|S| + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - k) \geq |T||S| + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - k)
\]

\[
= \sum_{x \in T} (|S| + d_{G-S}(x) - d_H(x) + e_H(x, S) - k) \geq \sum_{x \in T} (d_G(x) - d_H(x) + e_H(x, S) - k)
\]

\[
\geq \sum_{x \in T} (\delta(G) - m - k) \geq 0,
\]

which is a contradiction. □

According to Claim 2, we have \(T \neq \emptyset\). Thus, we may define \(h = \min\{d_{G-S}(x) - d_H(x) + e_H(x, S) \mid x \in T\}\). And let \(x_1\) be a vertex in \(T\) satisfying \(d_{G-S}(x_1) - d_H(x_1) + e_H(x_1, S) = h\). Then we have \(0 \leq h \leq k - 1\) according to the definition of \(T\) and \(d(x_1) \leq d_{G-S}(x_1) + |S| = h + d_H(x_1) - e_H(x_1, S) + |S|\).

In view of the condition of Theorem 5, the following inequalities hold:

\[
\frac{n}{2} \leq \delta(G) \leq d(x_1) \leq h + d_H(x_1) - e_H(x_1, S) + |S|,
\]

that is,

\[
|S| \geq \frac{n}{2} - (h + d_H(x_1) - e_H(x_1, S)).
\]

(2)

Now in order to prove the theorem, we shall deduce some contradictions in view of the following two cases:

**Case 1.** \(h = 0\).

By (1), (2) and \(|S| + |T| \leq n\), we get

\[
-1 \geq k|S| + \sum_{x \in T} (d_{G-S}(x) - d_H(x) + e_H(x, S) - k) \geq k|S| + h|T| - k|T| = k|S| - k|T|
\]

\[
\geq k|S| - k(n - |S|) = 2k|S| - kn \geq 2k\left(\frac{n}{2} - (d_H(x_1) - e_H(x_1, S))\right) - kn = -2k(d_H(x_1) - e_H(x_1, S),
\]

which implies \(d_H(x_1) - e_H(x_1, S) \geq \frac{1}{2k} > 0\).

According to the integrality of \(d_H(x_1) - e_H(x_1, S)\), we have \(d_H(x_1) - e_H(x_1, S) \geq 1\).

For some \(x \in T \setminus \{x_1\}\), if \(d_{G-S}(x) - d_H(x) + e_H(x, S) = 0\), then we similarly get \(d_H(x) - e_H(x, S) \geq 1\). Hence, one of (a) and (b) holds for any \(x \in T \setminus \{x_1\}\):
References

Theorem 5 is best possible.

Remark. Let us show that the condition 

\[ k \] 

is a contradiction.

\[ k \]

Thus, we have

\[ \sum_{x \in T} (d_G(x) - d_H(x) + \varepsilon(x,S)) \geq |T| - 2m. \]  

(3)

In view of (1), (2), (3), \( h = 0, |S| + |T| \leq n \) and \( n \geq 4k - 5 + 2(2k + 1)m \), we get

\[ -1 \geq k(S) + \sum_{x \in T} (d_G(x) - d_H(x) + \varepsilon(x,S)) \geq k|S| + |T| - 2m - k|T| \]

\[ = k|S| - (k - 1)|T| - 2m \geq k|S| - (k - 1)(n - |S|) - 2m = (k - 1)|S| - (k - 1)n - 2m \]

\[ \geq (2k - 1)(\frac{n}{2} - (d_H(x_1) - \varepsilon(x_1, S))) - (k - 1)n - 2m \geq (2k - 1)(\frac{n}{2} - m) - (k - 1)n - 2m \]

\[ = \frac{n}{2} - (2k + 1)m \geq \frac{4k - 5 + 2(2k + 1)m}{2} - (2k + 1)m > 2k - 3 \geq -1, \]

a contradiction.

Case 2. \( 1 \leq h \leq k - 1 \).

According to (1), (2), \( n \geq 4k - 5 + 2(2k + 1)m \) and \( |S| + |T| \leq n \), we obtain

\[ -1 \geq k(S) + \sum_{x \in T} (d_G(x) - d_H(x) + \varepsilon(x,S)) \geq k|S| + h|T| - k|T| \]

\[ = k|S| - (k - h)|T| \geq k|S| - (k - h)(n - |S|) = (k - h)|S| - (k - h)n \]

\[ \geq (2k - h)(\frac{n}{2} - (d_H(x_1) - \varepsilon(x_1, S))) - (k - h)n \]

\[ = \frac{hn}{2} - (2k - h)(h + d_H(x_1) - \varepsilon(x_1, S)) \geq \frac{hn}{2} - (2k - h)(h + m) \]

\[ \geq \frac{h(4k - 5 + 2(2k + 1)m)}{2} - (2k - h)(h + m) \]

\[ \geq \frac{h(4k - 6 + 2(2k + 1)m)}{2} - (2k - h)(h + m) = h^2 + 2(k + 1)m - 3h - 2km, \]

that is,

\[ -1 > h^2 + 2(k + 1)m - 3h - 2km. \]  

(4)

Let \( f(h) = h^2 + 2(k + 1)m - 3h - 2km \). Clearly, the function \( f(h) \) attains its minimum value at \( h = 1 \) since \( 1 \leq h \leq k - 1 \). Then we get \( f(h) \geq f(1) \). Combining this with (4) and \( m \geq 1 \), we have \( -1 > f(h) \geq f(1) = 2m - 2 \geq 0 \). It is a contradiction.

Completing the proof of Theorem 5. □

Remark. Let us show that the condition \( \delta(G) \geq \frac{n}{2} \) in Theorem 5 cannot be replaced by \( \delta(G) \geq \frac{n - 1}{2} \). Let \( G = K_{2k-3+(2k+1)m} \). Then we have \( n = 4k - 5 + 2(2k + 1)m \) and \( \delta(G) = 2k - 3 + (2k + 1)m = \frac{n - 1}{2} \). Let \( G' = G - E(mK_2) \), \( S = V(K_{2k-3+(2k+1)m}) \subseteq V(G) \) and \( T = V((2k - 1)m + 2k - 2)K_1 \cup (mK_2) \subseteq V(G) \), then \|S\| = 2k - 3 + (2k + 1)m, \|T\| = 2k - 2 + (2k + 1)m and \( d_{G'}(S,T) = 0 \). Thus, we get \( \delta(G', T) = k|S| + d_{G'}(T) - k|T| = k(2k - 3 + (2k + 1)m) - k(2k - 2 + (2k + 1)m) = -k < 0 \). By Lemma 2.1, \( G' \) has no fractional \( k \)-factor. Hence, \( G \) is not a fractional \( (k,m) \)-deleted graph. In the above sense, the result in Theorem 5 is best possible.

References