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COMPTES RENDUS MATHEMATIQUE

Ordinary Differential Equations

Existence of local solutions for the Boltzmann equation without angular cutoff

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Abstract

We consider the spatially inhomogeneous Boltzmann equation without angular cutoff. We prove the existence and uniqueness of local classical solutions to the Cauchy problem, in the function space with Maxwellian type exponential decay with respect to the velocity variable. ***To cite this article: R. Alexandre et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Existence locale pour l'équation de Boltzmann sans troncature. Nous considérons l'équation de Boltzmann inhomogène sans hypothèse de troncature angulaire. Nous montrons l'existence de solutions locales classiques ainsi que leur unicité pour le problème de Cauchy, dans une classe de fonctions exponentiellement décroissantes du type Maxwellian, relativement à la variable de vitesse. ***Pour citer cet article : R. Alexandre et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

On considère l'équation de Boltzmann inhomogène et non linéaire. Nous supposons que le noyau de collision B a une singularité en angle près de 0, et donc en particulier, nous ne faisons pas l'hypothèse de troncature angulaire de Grad. Le résultat de cette Note concerne l'existence d'une solution locale classique pour l'équation de Boltzmann. Elle fait suite à une Note précédente des auteurs [2], concernant les effets de régularisation pour cette même équation. Pour les hypothèses et les notations, on renvoit à la version anglaise.

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Théorème 0.1. *On suppose que* $0 < s < 1/2$ *et* $\gamma + 2s < 1$. *Soit* $f_0 \geqslant 0$ *et* $f_0 \in \mathcal{E}_0^{k_0}(\mathbb{R}^6)$ *pour un* $k_0 \geqslant 4$. *Alors il existe* $T_* > 0$ *tel que le problème de Cauchy*

$$\begin{cases} f_t + v \cdot \nabla_x f = Q(f, f), \\ f|_{t=0} = f_0, \end{cases}$$

admet une unique solution non-négative dans $\mathcal{E}^{k_0}([0, T_*] \times \mathbb{R}^6)$: *plus précisément, il existe* $\rho > 0$ *tel que*

$$e^{\rho \langle v \rangle^2} f \in C^0\big([0, T_*];\ H^{k_0}(\mathbb{R}^6)\big).$$

Si de plus on suppose que la donnée initiale $f_0 \in \mathcal{E}_0^5(\mathbb{R}^6)$ *et qu'elle ne s'annule pas sur un compact* $K \subset \mathbb{R}_x^3$, *c'est-à-dire*

$$\big\| f_0(x, \cdot)\big\|_{L^1(\mathbb{R}_v^3)} > 0, \quad \forall x \in K,$$

alors les solutions obtenues ci-dessus sont régulières près de K : *il existe* $0 < \tilde{T}_0 \leqslant T_*$ *et un voisinage* V_0 *de* K *dans* $\mathbb{R}_x^3$ *tels que*

$$f \in C^\infty\big(]0, \tilde{T}_0[\times V_0;\ \mathcal{S}(\mathbb{R}_v^3)\big).$$

Enfin, si $\gamma \leqslant 0$, *la solution non-négative du problème de Cauchy est unique dans* $C^0([0, T_*];\ H_p^m(\mathbb{R}^6))$ *pour* $m > 3/2 + 2s$, $p > 3/2 + 4s$.

Tous les détails de cette Note sont explicités dans la prépublication [3].

1. Introduction and main result

This Note is a continuation of [2], concerning the nonlinear spatially inhomogeneous and angular non-cutoff Boltzmann equation

$$f_t + v \cdot \nabla_x f = Q(f, f).$$

Above, $f = f(t, x, v)$ is the density distribution of particles, with position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$, at time $t \in]0, T[$ for some fixed $0 < T \leqslant +\infty$. The right-hand side of equation is given by the Boltzmann bilinear collision operator, acting only on the variable v,

$$Q(g, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma)\big\{g(v_*')f(v') - g(v_*)f(v)\big\}\, \mathrm{d}\sigma\, \mathrm{d}v_*,$$

for some suitable functions f and g. Here B is the non-cutoff cross-section, and for given $v, v_* \in \mathbb{R}^3$, for $\sigma \in \mathbb{S}^2$, the relations between the post- and pre-collisional velocities are given by

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \qquad v_*' = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma.$$

We assume that B is a non-negative function of the form

$$B\big(|v - v_*|, \cos\theta\big) = \Phi\big(|v - v_*|\big) b(\cos\theta), \qquad \cos\theta = \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}, \tag{1}$$

while the kinetic factor Φ and the angular factor b satisfy

$$\Phi\big(|v - v_*|\big) = \big(1 + |v - v_*|^2\big)^{\frac{\gamma}{2}}, \qquad b(\cos\theta) \approx K\theta^{-2-2s}, \quad \text{when } \theta \to 0+, \tag{2}$$

where $K > 0$ is a constant. Here, we assume Φ is regular near 0 to keep the calculation as simple as possible.

For the spatially homogeneous problem, that is, for the solutions independent of the space variable x, the existence theory is now well established, see [6,7]. However, there are very few results about the spatially inhomogeneous problem.

In this Note, we shall prove the existence and uniqueness of local classical solutions for the spatially inhomogeneous case so that it complements the regularization result shown in [2]. Let us recall that some solutions have already been constructed in [1,4]. However, they are not sufficient to initialize the regularization properties shown in [2], see also [5] for Gevrey class solutions.

We introduce the following weighted (with respect to the velocity variable $v \in \mathbb{R}^3$) Sobolev spaces. For $m \in \mathbb{R}$, $l \in \mathbb{R}$, set $W_l = \langle v\rangle^l = (1+|v|^2)^{l/2}$ and $H^m_l(\mathbb{R}^6) = \{f \in \mathcal{S}'(\mathbb{R}^6_{x,v});\ W_l f \in H^m(\mathbb{R}^6_{x,v})\}$. Furthermore, for $m \in \mathbb{R}$, we set

$$\mathcal{E}^m_0(\mathbb{R}^6) = \{g \in \mathcal{D}'(\mathbb{R}^6_{x,v});\ \exists \rho_0 > 0 \text{ s.t. } e^{\rho_0\langle v\rangle^2} g \in H^m(\mathbb{R}^6_{x,v})\},$$

and for $T > 0$

$$\mathcal{E}^m([0,T]\times\mathbb{R}^6_{x,v}) = \{f \in C^0([0,T];\ \mathcal{D}'(\mathbb{R}^6_{x,v}));\ \exists \rho > 0 \text{ s.t. } e^{\rho\langle v\rangle^2} f \in C^0([0,T];\ H^m(\mathbb{R}^6_{x,v}))\}.$$

Theorem 1.1. *Assume that $0 < s < 1/2$ and $\gamma + 2s < 1$. Let $f_0 \geqslant 0$ and $f_0 \in \mathcal{E}^{k_0}_0(\mathbb{R}^6)$ for some $k_0 \geqslant 4$. Then there exists $T_* > 0$ such that the Cauchy problem*

$$\begin{cases} f_t + v\cdot\nabla_x f = Q(f,f), \\ f|_{t=0} = f_0, \end{cases} \tag{3}$$

admits a unique non-negative solution in the function space $\mathcal{E}^{k_0}([0,T_*]\times\mathbb{R}^6)$. Precisely, there exists $\rho > 0$ such that

$$e^{\rho\langle v\rangle^2} f \in C^0([0,T_*];\ H^{k_0}(\mathbb{R}^6)).$$

Furthermore, if we assume that the initial data f_0 is in $\mathcal{E}^5_0(\mathbb{R}^6)$ and does not vanish on a compact set $K \subset \mathbb{R}^3_x$, that is,

$$\|f_0(x,\cdot)\|_{L^1(\mathbb{R}^3_v)} > 0, \quad \forall x \in K,$$

then we have the full regularity near K of the above solution, that is, there exist $0 < \tilde{T}_0 \leqslant T_*$ and a neighborhood V_0 of K in $\mathbb{R}^3_x$ such that

$$f \in C^\infty(]0,\tilde{T}_0[\times V_0;\ \mathcal{S}(\mathbb{R}^3_v)).$$

Moreover, if $\gamma \leqslant 0$, the non-negative solution of the Cauchy problem (3) *is unique in the function space $C^0([0,T_*];\ H^m_p(\mathbb{R}^6))$ for $m > 3/2 + 2s$, $p > 3/2 + 4s$.*

Detailed proofs together with more explanations are available in [3]. In particular, the lifetime T_* is given therein more precisely. We believe in fact that global existence is true for solutions close to Maxwellians, and hope to report soon on this point.

2. Ideas of proof

The first main step is to consider a modified Cauchy problem. For fixed $\kappa, \rho > 0$, we set, for $0 \leqslant t \leqslant T_0 := \rho/(2\kappa)$

$$\mu_\kappa(t) = \mu(t,v) = e^{-(\rho-\kappa t)\langle v\rangle^2}$$

and

$$f = \mu_\kappa(t) g, \qquad \Gamma^t(g,g) = \mu_\kappa(t)^{-1} Q(\mu_\kappa(t)g,\ \mu_\kappa(t)g).$$

Then the Cauchy problem (3) is rewritten as

$$\begin{cases} g_t + v\cdot\nabla_x g + \kappa\langle v\rangle^2 g = \Gamma^t(g,g), \\ g|_{t=0} = g_0, \end{cases} \tag{4}$$

for which one has

Theorem 2.1. *Assume that $0 < s < 1/2$, $\gamma + 2s < 1$ and $\kappa, \rho > 0$. Let $g_0 \in H^k_l(\mathbb{R}^6)$, $g_0 \geqslant 0$ for some $l \geqslant 3$ and $k \geqslant 4$. Then there exists $T_* \in]0, T_0]$ such that the Cauchy problem* (4) *admits a unique non-negative solution*

$$g \in C^0([0,T_*];\ H^k_l(\mathbb{R}^6)) \cap L^2(]0,T_*[;\ H^k_{l+1}(\mathbb{R}^6)).$$

The proof of Theorem 2.1 is based on cut-off approximations. For $0 < \varepsilon \ll 1$, we set $b_\varepsilon(\cos\theta) = b(\cos\theta)$ if $|\theta| \geqslant 2\varepsilon$, and $b(\cos\varepsilon)$ if $|\theta| \leqslant \varepsilon$. Then, let $\Gamma_\varepsilon^t(g,g)$ be the collision operator defined by the cut-off kernel $B_\varepsilon = \Phi(v-v_*)b_\varepsilon(\cos\theta)$. It follows that, for suitable functions U, V,

$$\Gamma_\varepsilon^t(U,V)(v) = \mu^{-1}(t,v) \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} B_\varepsilon(v-v_*,\sigma)\big(\mu'_*(t)U'_*\mu'(t)V' - \mu_*(t)U_*\mu(t)V\big)\,\mathrm{d}v_*\,\mathrm{d}\sigma$$
$$= Q_\varepsilon\big(\mu(t)U,V\big) + \int_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} B_\varepsilon(v-v_*,\sigma)\big(\mu_*(t)-\mu'_*(t)\big)U'_*V'\,\mathrm{d}v_*\,\mathrm{d}\sigma,$$

which satisfies the following weighted upper bound estimate:

Lemma 2.2. *Let $\gamma \in \mathbb{R}$. Then for any $\varepsilon > 0$, $k \geqslant 4$, $l \geqslant 0$, there exists $C > 0$ depending on ε, k, l such that for any U, V belonging to $H^k_l(\mathbb{R}^6)$,*

$$\big\|\Gamma_\varepsilon^t(U,V)\big\|_{H^k_l(\mathbb{R}^6)} \leqslant C\|U\|_{H^k_{l+\gamma^+}(\mathbb{R}^6)}\|V\|_{H^k_{l+\gamma^+}(\mathbb{R}^6)}, \quad 0 \leqslant t \leqslant T_0 = \frac{\rho}{2\kappa}.$$

As regards to the Cauchy problem

$$\begin{cases} g_t + v\cdot\nabla_x g + \kappa\langle v\rangle^2 g = \Gamma_\varepsilon^t(g,g), \\ g|_{t=0} = g_0, \end{cases} \tag{5}$$

we have the following existence result:

Proposition 2.1. *Assume that $\gamma \leqslant 1$. Let $k \geqslant 4$, $l \geqslant 0$, $\varepsilon > 0$ and $D_0 > 0$. Then, there exists $T_\epsilon \in]0, T_0]$ such that for any non-negative initial data g_0 satisfying*

$$g_0 \in H^k_l\big(\mathbb{R}^6\big), \qquad \|g_0\|_{H^k_l(\mathbb{R}^6)} \leqslant D_0,$$

the Cauchy problem (5) *admits a unique non-negative solution g^ε having the property*

$$g^\varepsilon \in C^0\big(]0,T_\epsilon[;\ H^k_l\big(\mathbb{R}^6\big)\big) \cap L^2\big(]0,T_\epsilon[;\ H^k_{l+1}\big(\mathbb{R}^6\big)\big), \qquad \big\|g^\varepsilon\big\|_{L^\infty(]0,T_\epsilon[;\ H^k_l(\mathbb{R}^6))} \leqslant 2D_0.$$

This existence result is proven by successive approximations that preserve non-negativity, which are defined by using the usual splitting of the cutoff collision operator into the gain (+) and loss (−) terms, and a careful use of Lemma 2.2. More precisely, we now define a sequence of approximate solutions (in mild formulation sense) $\{g^n\}_{n\in\mathbb{N}}$ by

$$\begin{cases} g^0 = g_0; \\ \partial_t g^{n+1} + v\cdot\nabla_x g^{n+1} + \kappa\langle v\rangle^2 g^{n+1} = \Gamma_\varepsilon^{t,+}\big(g^n, g^n\big) - \Gamma_\varepsilon^{t,-}\big(g^n, g^{n+1}\big), \\ g^{n+1}|_{t=0} = g_0. \end{cases} \tag{6}$$

Lemma 2.2 implies that, for any $T \in]0, T_0]$, $g_0 \geqslant 0$, $g^n \in L^\infty(]0,T[;\ H^k_l(\mathbb{R}^6))$, $g^n \geqslant 0$, there exists a non-negative solution g^{n+1}, but with a possible weight loss in v. The point is that the term $\kappa\langle v\rangle^2 g^{n+1}$ in (6) not only recovers this weight loss but also creates a higher moment. Introduce the space and norm

$$X = L^\infty\big(]0,T[;\ H^k_l\big(\mathbb{R}^6\big)\big) \cap L^2\big(]0,T[;\ H^k_{l+1}\big(\mathbb{R}^6\big)\big),$$
$$|||g|||^2 = \|g\|^2_{L^\infty(]0,T[;\ H^k_l(\mathbb{R}^6))} + \kappa\|g\|^2_{L^2(]0,T[;\ H^k_{l+1}(\mathbb{R}^6))}.$$

Lemma 2.3. *Assume $\gamma \leqslant 1$ and let $k \geqslant 4$, $l \geqslant 0$, $\varepsilon > 0$. Then, there exist positive constants C_1, C_2 such that if ρ, $\kappa > 0$ and if*

$$g_0 \in H^k_l\big(\mathbb{R}^6\big), \qquad g^n \in L^\infty\big(]0,T[;\ H^k_l\big(\mathbb{R}^6\big)\big),$$

with some $T \leqslant T_0$, the function g^{n+1} given by (6) *enjoys the properties*

$$g^{n+1} \in X, \qquad |||g^{n+1}|||^2 \leqslant e^{C_1 K_n T}\left(\|g_0\|^2_{H^k_l(\mathbb{R}^6)} + \frac{C_2}{\kappa}\|g^n\|^4_{L^4(]0,T[;\ H^k_l(\mathbb{R}^6))}\right),$$

where K_n is a positive constant depending on $\|g^n\|_{L^\infty(]0,T[;\ H^k_l(\mathbb{R}^6))}$ and κ.

This lemma ensures a uniform bound of the sequence $\{g^n\}$, and then leads to Proposition 2.1 on the existence of a unique solution g^ε for the cutoff Cauchy problem (5) for each fixed $\varepsilon > 0$.

The next step is the uniform estimation with respect to ε.

Proposition 2.2. *Assume that $0 < s < 1/2$, $\gamma + 2s < 1$. Let $g_0 \in H^k_l(\mathbb{R}^6)$, $g_0 \geqslant 0$ for some $k \geqslant 4$, $l \geqslant 3$. Then there exists $T_* \in]0, T_0]$ depending only on $\|g_0\|_{H^k_l}$ and independent of ε such that if for some $0 < T \leqslant T_0$,*

$$g^\varepsilon \in C^0\big([0,T];\ H^k_l\big(\mathbb{R}^6\big)\big) \cap L^2\big(]0,T[;\ H^k_{l+1}\big(\mathbb{R}^6\big)\big)$$

is a non-negative solution of the Cauchy problem (5)*, then it holds that*

$$\big\|g^\varepsilon\big\|_{L^\infty(]0,T_{**}[;\ H^k_l(\mathbb{R}^6))} \leqslant 2\|g_0\|_{H^k_l(\mathbb{R}^6)},$$

for $T_{**} = \min\{T, T_*\}$.

The proof of this proposition uses in an essential way the results in [3] on the coercivity and upper bound estimates for non-cutoff collision operators.

Combining Propositions 2.1 and 2.2, we can extend the solution g^ε to the fixed interval $[0, T_*]$ for any $0 < \varepsilon < 1$.

Proposition 2.3. *Assume that $0 < s < 1/2$, $\gamma + 2s < 1$, $g_0 \geqslant 0$, $g_0 \in H^k_l(\mathbb{R}^6)$ for some $k \geqslant 4$, $l \geqslant 3$. Let $T_* > 0$ be given in Proposition* 2.2*. Then the Cauchy problem* (5) *admits a unique non-negative solution up to T_* satisfying*

$$g^\varepsilon \in C^0\big([0,T_*];\ H^k_l\big(\mathbb{R}^6\big)\big) \cap L^2\big(]0,T_*[;\ H^k_{l+1}\big(\mathbb{R}^6\big)\big).$$

By combining the previous estimates together, the existence part proof of Theorem 2.1 is completed by showing the convergence of the sequence $\{g^\varepsilon\}$. The uniqueness of solution is shown by the coercivity and upper bound estimates for collision operators.

Sketch of proof of Theorem 1.1. To prove the existence, let $f_0 \in \mathcal{E}^{k_0}_0(\mathbb{R}^6)$. Then there exists $\rho_0 > 0$ such that $e^{\rho_0\langle v\rangle^2} f_0 \in H^{k_0}(\mathbb{R}^6)$. Choose $0 < \rho < \rho_0$, $\kappa > 0$ and set $g_0 = e^{\rho\langle v\rangle^2} f_0$. If g is a non-negative solution given in Theorem 2.1 for the Cauchy problem (4) with this g_0,

$$f(t,x,v) = e^{-(\rho-\kappa t)\langle v\rangle^2} g(t,x,v) \in C^0\big([0,T_*];\ H^{k_0}_l\big(\mathbb{R}^6\big)\big) \quad \forall l \in \mathbb{N},$$

is a non-negative solution of the Cauchy problem (3). To prove the uniqueness, suppose that there exist two solutions $f_1 \in \mathcal{E}^{k_0}([0,T_1] \times \mathbb{R}^6_{x,v})$ and $f_2 \in \mathcal{E}^{k_0}([0,T_2] \times \mathbb{R}^6_{x,v})$. This implies that there exist $\rho_1, \rho_2 > 0$ such that

$$e^{\rho_1\langle v\rangle^2} f_1 \in C^0\big([0,T_1];\ H^{k_0}\big(\mathbb{R}^6\big)\big), \qquad e^{\rho_2\langle v\rangle^2} f_2 \in C^0\big([0,T_2];\ H^{k_0}\big(\mathbb{R}^6\big)\big).$$

Take $0 < \rho < \min\{\rho_0, \rho_1, \rho_2\}$ and $\kappa > 0$ sufficiently small such that $\frac{\rho}{2\kappa} > T_* := \min\{T_1, T_2\}$. Then

$$g_1 = e^{(\rho-\kappa t)\langle v\rangle^2} f_1, \qquad g_2 = e^{(\rho-\kappa t)\langle v\rangle^2} f_2 \in C^0\big([0,T_*];\ H^{k_0}_l\big(\mathbb{R}^6\big)\big)$$

are two solutions of the Cauchy problem (4) with the common initial datum g_0. The uniqueness of Theorem 2.1 implies $g_1 = g_2$, so that $f_1 = f_2$ for $t \in [0, T_*]$. The regularizing effect follows from Theorem 1.1 of [2].

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