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Probability Theory

On the Borel–Cantelli lemma and its generalization

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Abstract

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of events on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We show that if $\lim_{m\to\infty}\sum_{n=1}^{m} w_n\mathbf{P}(A_n) = \infty$ where each $w_n \in \mathbb{R}$, then

$$\mathbf{P}(\limsup A_n) \geqslant \limsup_{n\to\infty} \frac{(\sum_{k=1}^{n} w_k\mathbf{P}(A_k))^2}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \mathbf{P}(A_i \cap A_j)}.$$

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Résumé

Sur le lemme de Borel–Cantelli et sa généralisation. Soit $\{A_n\}_{n=1}^{\infty}$ une séquence d'événements dans un éspace de probabilité $(\Omega, \mathcal{F}, \mathbf{P})$. On montre que, si $\lim_{m\to\infty}\sum_{n=1}^{m} w_n\mathbf{P}(A_n) = \infty$ où chaque $w_n \in \mathbb{R}$, alors

$$\mathbf{P}(\limsup A_n) \geqslant \limsup_{n\to\infty} \frac{(\sum_{k=1}^{n} w_k\mathbf{P}(A_k))^2}{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \mathbf{P}(A_i \cap A_j)}.$$

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1. Introduction

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of events on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The classical Borel–Cantelli lemma states that: (a) if $\sum_{n=1}^{\infty}\mathbf{P}(A_n) < \infty$, then $\mathbf{P}(\limsup A_n) = 0$; (b) if $\sum_{n=1}^{\infty}\mathbf{P}(A_n) = \infty$ and $\{A_n\}_{n=1}^{\infty}$ are mutually independent, then $\mathbf{P}(\limsup A_n) = 1$. Here $\limsup A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k$. The Borel–Cantelli lemma played an exceptionally important role in probability theory. Many investigations were devoted to the second part of the Borel–Cantelli lemma in the attempt to weaken the independence condition on $\{A_n\}_{n=1}^{\infty}$.

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Erdös and Rényi [4] proved that the mutual independence condition on $\{A_n\}_{n=1}^{\infty}$ can be replaced by the weaker condition of pairwise independence. Indeed they [8] (see also [3,5,9]) proved a more general theorem: if $\sum_{n=1}^{\infty}\mathbf{P}(A_n)=\infty$, then

$$\mathbf{P}(\limsup A_n)\geqslant \limsup_{n\to\infty}\frac{(\sum_{k=1}^{n}\mathbf{P}(A_k))^2}{\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{P}(A_i\cap A_j)}.$$

There are many discussions and generalizations towards the Borel–Cantelli lemma, for example see [1,6,7,10]. The main purpose of this paper is to present a weighted version of the Erdös–Rényi theorem:

Theorem 1. *Suppose* $\lim_{m\to\infty}\sum_{n=1}^{m}w_n\mathbf{P}(A_n)=\infty$*, where each* w_n *is a real weight* (*which could be negative*)*. Then*

$$\mathbf{P}(\limsup A_n)\geqslant \limsup_{n\to\infty}\frac{(\sum_{k=1}^{n}w_k\mathbf{P}(A_k))^2}{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j\mathbf{P}(A_i\cap A_j)}.$$

The proof of Theorem 1 is relatively easy if we further assume all terms of the weight sequence to be nonnegative. By choosing each $w_n=1/\mathbf{P}(A_n)$ in Theorem 1, we obtain the following corollary:

Corollary 2. *Suppose* $\mathbf{P}(A_n)>0$ *holds for all* $n\in\mathbb{N}$*. Then*

$$\mathbf{P}(\limsup A_n)\geqslant \limsup_{n\to\infty}\frac{n^2}{\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\mathbf{P}(A_i\cap A_j)}{\mathbf{P}(A_i)\mathbf{P}(A_j)}}.$$

2. Proof of the main result

For any matrix $E=(z_{ij})_{m\times n}$, denote by $\Gamma(E)$ the sum of all its entries, that is, $\Gamma(E)=\sum_{i=1}^{m}\sum_{j=1}^{n}z_{ij}$.

Lemma 3. *Given a partition of an* $(m+n)\times(m+n)$ *symmetric matrix* E:

$$E=\begin{pmatrix}A_{m\times m} & C_{m\times n}\\ C^T_{m\times n} & B_{n\times n}\end{pmatrix}.$$

If E *is positive semi-definite, then* $\Gamma(C)^2\leqslant\Gamma(A)\Gamma(B)$.

Proof. This lemma follows from the following inequality: $\forall x,y\in\mathbb{R}$,

$$(x,\dots,x,y,\dots,y)E(x,\dots,x,y,\dots,y)^T=\Gamma(A)x^2+2\Gamma(C)xy+\Gamma(B)y^2\geqslant 0.\qquad\square$$

Lemma 4. *Let* $\{A_i\}_{i=1}^{n}$ *be finitely many events on a probability space* $(\Omega,\mathcal{F},\mathbf{P})$*. Then the matrix* $(\mathbf{P}(A_i\cap A_j))_{n\times n}$ *is positive semi-definite.*

Proof. Let $\mathbb{E}(\cdot)$ be the expectation function and χ_{A_i} be the indicator function of the event A_i. Then $\mathbf{P}(A_i)=\mathbb{E}(\chi_{A_i})$ and $\mathbf{P}(A_i\cap A_j)=\mathbb{E}(\chi_{A_i}\chi_{A_j})$. For each $(s_1,s_2,\dots,s_n)\in\mathbb{R}^n$,

$$\begin{aligned}(s_1,s_2,\dots,s_n)\big(\mathbf{P}(A_i\cap A_j)\big)(s_1,s_2,\dots,s_n)^T&=\sum_{i=1}^{n}\sum_{j=1}^{n}s_is_j\mathbf{P}(A_i\cap A_j)=\sum_{i=1}^{n}\sum_{j=1}^{n}s_is_j\mathbb{E}(\chi_{A_i}\chi_{A_j})\\&=\mathbb{E}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}s_is_j\chi_{A_i}\chi_{A_j}\right)=\mathbb{E}\left(\left(\sum_{i=1}^{n}s_i\chi_{A_i}\right)^2\right)\geqslant 0.\end{aligned}$$

This proves the lemma. $\square$

Lemma 5. *Suppose* $\lim_{m\to\infty}\sum_{n=1}^{m}w_n\mathbf{P}(A_n)=\infty$*, where each* $w_n\in\mathbb{R}$*. Then*

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j\mathbf{P}(A_i\cap A_j)}{\sum_{i=2}^{n}\sum_{j=2}^{n}w_iw_j\mathbf{P}(A_i\cap A_j)}=1.$$

Proof. By Lemma 4, $E_n \doteq (w_i w_j \mathbf{P}(A_i \cap A_j))_{n\times n}$ is positive semi-definite. Define

$$A = \big(w_1 w_1 \mathbf{P}(A_1 \cap A_1)\big), \quad B_n = \big(w_i w_j \mathbf{P}(A_i \cap A_j)\big)_{2\leqslant i,j\leqslant n}, \quad \text{and} \quad C_n = \big(w_1 w_j \mathbf{P}(A_1 \cap A_j)\big)_{2\leqslant j\leqslant n}.$$

By Lemma 3, $\Gamma(C_n)^2 \leqslant \Gamma(A)\Gamma(B_n)$ ($\forall n \geqslant 2$). By the Cauchy–Schwarz inequality,

$$\left(\sum_{i=2}^n w_i \mathbf{P}(A_i)\right)^2 = \left(\mathbb{E}\left(\sum_{i=2}^n w_i \chi_{A_i}\right)\right)^2 \leqslant \mathbf{P}\left(\bigcup_{i=2}^n A_i\right)\cdot \mathbb{E}\left(\left(\sum_{i=2}^n w_i \chi_{A_i}\right)^2\right)$$
$$= \mathbf{P}\left(\bigcup_{i=2}^n A_i\right)\cdot\left(\sum_{i=2}^n\sum_{j=2}^n w_i w_j \mathbf{P}(A_i\cap A_j)\right) \leqslant \Gamma(B_n).$$

Since $\lim_{n\to\infty}\sum_{i=2}^n w_i \mathbf{P}(A_i) = \infty$, we have $\Gamma(B_n)\to\infty$ as n approaches to infinity. Hence

$$\lim_{n\to\infty}\frac{\Gamma(A)+\Gamma(B_n)+2\Gamma(C_n)}{\Gamma(B_n)} = 1 + \lim_{n\to\infty}\frac{2\Gamma(C_n)}{\Gamma(B_n)} = 1.$$

This proves the lemma. □

Remark 1. We obtained the following by-product from the proof of the above lemma:

$$\left(\sum_{i=1}^n w_i \mathbf{P}(A_i)\right)^2 \leqslant \mathbf{P}\left(\bigcup_{i=1}^n A_i\right)\cdot\left(\sum_{i=1}^n\sum_{j=1}^n w_i w_j \mathbf{P}(A_i\cap A_j)\right). \tag{1}$$

This formula can be viewed as a weighted version of the Chung–Erdös inequality [2].

Proposition 6. *Suppose* $\lim_{m\to\infty}\sum_{n=1}^m w_n \mathbf{P}(A_n) = \infty$, *where each* $w_n \in \mathbb{R}$. *Then for all* $s \in \mathbb{N}$,

$$\limsup_{n\to\infty}\frac{(\sum_{k=1}^n w_k \mathbf{P}(A_k))^2}{\sum_{i=1}^n\sum_{j=1}^n w_i w_j \mathbf{P}(A_i\cap A_j)} = \limsup_{n\to\infty}\frac{(\sum_{k=s}^n w_k \mathbf{P}(A_k))^2}{\sum_{i=s}^n\sum_{j=s}^n w_i w_j \mathbf{P}(A_i\cap A_j)}.$$

Proposition 6 is an immediate corollary of Lemma 5. With all the above preparation in hand, we are ready to prove Theorem 1.

Proof of Theorem 1. By (1) and Proposition 6,

$$\mathbf{P}(\limsup A_n) = \mathbf{P}\left(\bigcap_{s=1}^\infty\bigcup_{k=s}^\infty A_k\right) = \lim_{s\to\infty}\mathbf{P}\left(\bigcup_{k=s}^\infty A_k\right) = \lim_{s\to\infty}\left(\lim_{n\to\infty}\mathbf{P}\left(\bigcup_{k=s}^n A_k\right)\right)$$
$$\geqslant \lim_{s\to\infty}\left(\limsup_{n\to\infty}\frac{(\sum_{k=s}^n w_k \mathbf{P}(A_k))^2}{\sum_{i=s}^n\sum_{j=s}^n w_i w_j \mathbf{P}(A_i\cap A_j)}\right) = \limsup_{n\to\infty}\frac{(\sum_{k=1}^n w_k \mathbf{P}(A_k))^2}{\sum_{i=1}^n\sum_{j=1}^n w_i w_j \mathbf{P}(A_i\cap A_j)}.$$

This completes the proof of Theorem 1. □

Example 1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, $A, B \in \mathcal{F}$, $\mathbf{P}(A\cap B) > 0$. For all $n \in \mathbb{N}$, let $A_{3n-2} = A$, $A_{3n-1} = B$, $A_{3n} = A\cap B$. By the Erdös–Rényi theorem,

$$\mathbf{P}(\limsup A_n) \geqslant \frac{(\mathbf{P}(A)+\mathbf{P}(B)+\mathbf{P}(A\cap B))^2}{\mathbf{P}(A)+\mathbf{P}(B)+7\mathbf{P}(A\cap B)}.$$

Applying Theorem 1 with the weight sequence $1, 1, -1, 1, 1, -1, 1, 1, -1, \ldots$, we obtain

$$\mathbf{P}(\limsup A_n) \geqslant \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A\cap B) = \mathbf{P}(A\cup B).$$

In fact $\mathbf{P}(\limsup A_n) = \mathbf{P}(A\cup B)$. So Theorem 1 is best possible for this example.

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