Algebraic Geometry

On the vector bundles over rationally connected varieties

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Received 28 June 2009; accepted after revision 2 September 2009
Available online 19 September 2009
Presented by Jean-Pierre Demailly

Abstract

Let $X$ be a rationally connected smooth projective variety defined over $\mathbb{C}$ and $E \to X$ a vector bundle such that for every morphism $\gamma : \mathbb{C}P^1 \to X$, the pullback $\gamma^* E$ is trivial. We prove that $E$ is trivial. Using this we show that if $\gamma^* E$ is isomorphic to $L(\gamma) \oplus r$ for all $\gamma$ of the above type, where $L(\gamma) \to \mathbb{C}P^1$ is some line bundle, then there is a line bundle $\xi$ over $X$ such that $E = \xi \oplus r$.

1. Introduction

Let $E$ be a holomorphic vector bundle over a connected complex projective manifold $X$. If for every pair of the form $(C, \gamma)$, where $C$ is a compact connected Riemann surface, and $\gamma : C \to X$ is a holomorphic map, the pullback $\gamma^* E$ is semistable, then it is known that $E$ is semistable, and $c_i(\text{End}(E)) = 0$ for all $i \geq 1$ [3, pp. 3–4, Theorem 1.2]. Our aim here is to show that if $X$ is rationally connected, then the above conclusion remains valid even if we insert in the condition that $C$ is a rational curve. We recall that a complex projective variety $X$ is said to be \textit{rationally connected} if any two points of $X$ can be joined by an irreducible rational curve on $X$; see [9, Theorem 2.1] for equivalent conditions. We prove the following theorem:

\textbf{Theorem 1.1.} Let $E$ be a vector bundle of rank $r$ over a rationally connected smooth projective variety $X$ defined over $\mathbb{C}$ such that for every morphism $\gamma : \mathbb{C}P^1 \to X$, the pullback $\gamma^* E$ is trivial. We then have $E = \xi \oplus r$. To cite this article: I. Biswas, J.P.P. dos Santos, C. R. Acad. Sci. Paris, Ser. I 347 (2009).

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Résumé

Des fibrés vectoriels sur les variétés rationnellement connexes. Soit $X$ une variété rationnellement connexe sur $\mathbb{C}$ et soit $E \to X$ un fibré vectoriel tel que, pour tout morphisme $\gamma : \mathbb{C}P^1 \to X$, le fibré $\gamma^* E$ est trivial. Nous montrons que $E$ est trivial. Nous en déduisons que si, pour tout $\gamma$ comme avant, $\gamma^* E$ est isomorphe à $L(\gamma)^{\oplus r}$, où $L(\gamma) \to \mathbb{C}P^1$ est un fibré en droites, alors il existe un fibré en droites $\xi$ sur $X$ et un isomorphisme $E \cong \xi^{\oplus r}$. Pour citer cet article : I. Biswas, J.P.P. dos Santos, C. R. Acad. Sci. Paris, Ser. I 347 (2009).

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\( \gamma : \mathbb{C}P^1 \rightarrow X, \)

the pullback \( \gamma^* E \) is isomorphic to \( L(\gamma)^{\oplus r} \) for some line bundle \( L(\gamma) \rightarrow \mathbb{C}P^1 \). Then there is a line bundle \( \xi \) over \( X \) such that \( E = \xi^{\oplus r} \).

In [1] this was proved under the extra assumption that \( \text{Pic}(X) = \mathbb{Z} \) (see [1, p. 211, Proposition 1.2]).

Theorem 1.1 is deduced from the following proposition (see Proposition 2.1):

**Proposition 1.2.** Let \( X \) be as in Theorem 1.1. Let \( E \rightarrow X \) be a vector bundle such that for every morphism \( \gamma : \mathbb{C}P^1 \rightarrow X \), the pullback \( \gamma^* E \) is trivial. Then \( E \) itself is trivial.

The condition in Theorem 1.1 that \( \gamma^* E \) is of the form \( L(\gamma)^{\oplus r} \) can be replaced by an equivalent condition which says that \( \gamma^* E \) is semistable (see Corollary 2.3).

2. Criterion for triviality

Let \( X \) be a rationally connected smooth projective variety defined over \( \mathbb{C} \). Let \( E \rightarrow X \) be a vector bundle.

**Proposition 2.1.** Assume that for every morphism

\[ \gamma : \mathbb{C}P^1 \rightarrow X \]

the vector bundle \( \gamma^* E \rightarrow \mathbb{C}P^1 \) is trivial. Then \( E \) itself is trivial.

**Proof.** Let \( x \in X \) be a closed point. There is a smooth family of rational curves on \( X \)

\[
\begin{array}{c}
Z \\
\sigma
\end{array} \xrightarrow{\phi} X
\]

(1)

where

1. \( T \) is open in \( \text{Mor}(\mathbb{C}P^1, X; (0 : 1) \mapsto x) \) (hence \( T \) is quasiprojective),
2. \( f \circ \sigma = \text{Id}_T \),
3. \( \phi \) is dominant, and
4. \( \phi(\sigma(t)) = x \) for all \( t \in T \).

(See [4, Section 3], [8, Theorem 3].)

Let

\[ \beta := [\phi(f^{-1}(t))] \in H_2(X, \mathbb{Z}) \]

be the homology class, where \( t \in T(\mathbb{C}) \). Let \( \overline{M}_{0,1}(X, \beta) \) be the moduli stack classifying families of stable maps from 1-pointed genus zero curves to \( X \) which represent the class \( \beta \). (We are following the terminology of [5].) We know that \( \overline{M}_{0,1}(X, \beta) \) is a proper Deligne–Mumford stack [2, p. 27, Theorem 3.14].

Let

\[
\rho : T \rightarrow \overline{M}_{0,1}(X, \beta)
\]

(2)

be the morphism associated to the family in (1).

By “Chow’s Lemma” [10, p. 154, Corollaire 16.6.1], there exists a projective \( \mathbb{C} \)-scheme \( Y \) together with a proper surjective morphism \( \psi : Y \rightarrow \overline{M}_{0,1}(X, \beta) \). There exists a Cartesian diagram

\[
\begin{array}{c}
T_1 \xrightarrow{\rho_1} Y \\
\psi_1 \\
T \xrightarrow{\rho} \overline{M}_{0,1}(X, \beta)
\end{array}
\]
where \( T_1 \) is a scheme and \( \psi_1 \) is proper and surjective. This last assertion is justified by the fact that the diagonal of a Deligne–Mumford stack is schematic ([10, p. 26, Lemme 4.2] and [10, p. 21, Corollaire 3.13]). As \( T \) is separated (it is open in \( \text{Mor}(\mathbb{CP}^1, X) \)), we can apply Nagata’s Theorem [11, p. 106, Théorème 3.2] to find a proper \( \mathbb{C} \)-scheme \( \overline{T}_1 \) and a schematically dense open immersion \( i : T_1 \hookrightarrow \overline{T}_1 \). Eliminating the “indeterminacy locus” (see e.g. [11, pp. 99–100]), we can find a blow-up

\[
\xi : \overline{T} \twoheadrightarrow \overline{T}_1
\]

whose center is disjoint from \( T_1 \) and a morphism

\[
\phi : \overline{T} \rightarrow Y
\]

which extends \( \rho_1 : T_1 \rightarrow Y \). The composition \( \psi \circ \phi : \overline{T} \rightarrow \overline{\mathcal{M}}_{0,1}(X, \beta) \) represents a family of 1-pointed genus zero stable maps

\[
\begin{array}{ccc}
\overline{T} & \xrightarrow{\phi} & X \\
\Downarrow \rho & & \\
\overline{T}_1 & \xrightarrow{\phi_1} & X
\end{array}
\]

(3)

whose pullback via \( i : T_1 \hookrightarrow \overline{T} \) is the pullback of the family in (1) via \( \psi_1 \). Clearly \( \phi \) is dominant (hence surjective) and \( \phi \circ \sigma \) is a constant morphism. Note that, without loss of generality, we can assume \( \overline{T} \) to be reduced.

We recall that the pullback of \( E \) by any map from \( \mathbb{CP}^1 \) is trivial. Consequently, for any point \( t \in \overline{T}(\mathbb{C}) \), the restriction of \( \overline{E} := \phi^*E \) to the curve \( \overline{T}^{-1}(t) \) — which is a tree of \( \mathbb{CP}^1 \) — is trivial. Therefore, \( \overline{E} \) descends to \( \overline{T} \). More precisely, the direct image \( \overline{f}_* \overline{E} \) is a vector bundle on \( \overline{T} \), and the canonical arrow

\[
\overline{f}_* \overline{E} \longrightarrow E
\]

(4)

is an isomorphism [12, §5]. The homomorphism in (4) is injective because any section of a trivial vector bundle, over a connected projective scheme, that vanishes at one point actually vanishes identically; the homomorphism is surjective also because \( \overline{E}|_{\overline{T}^{-1}(t)} \) is trivial for all \( t \). We also note that the image of (4) by \( \sigma^* \) defines an isomorphism between \( \sigma^* \overline{E} \) and \( \overline{f}_* \overline{E} \). Therefore, using (4),

\[
\overline{f}_* \sigma^* \overline{E} = E.
\]

(5)

Now from the condition that \( \phi \circ \sigma \) is a constant map it follows immediately that \( \sigma^* \overline{E} = \sigma^* \overline{E} \) is a trivial vector bundle. Consequently, using (5) we conclude that the vector bundle \( \phi^*E \) is trivial.

Since \( \phi \) is a surjective and proper morphism, and \( \phi^*E \) is trivial, we conclude that the Chern class \( c_i(E) \) is numerically equivalent to zero for all \( i \geq 1 \).

Next we will show that the vector bundle \( E \) is semistable.

Let \( C \hookrightarrow X \) be a smooth irreducible (proper) curve on \( X \), and let \( C' \hookrightarrow \overline{Z} \) be an irreducible curve such that \( \phi(C') = C \). (The curve \( C' \) can be constructed as the closure of a closed point of the generic fiber of \( \phi^{-1}(C) \twoheadrightarrow C \).)

Since the pullback of \( E|_C \) to \( C' \) is trivial, so is the pullback of \( E|_C \) to the normalization of \( C' \). Consequently, the vector bundle \( E|_C \) is semistable of degree zero. This allows us to conclude that \( E \) is semistable with respect to any chosen polarization on \( X \).

Since \( E \) is semistable, and both \( c_1(E) \) and \( c_2(E) \) are numerically equivalent to zero, a theorem of Simpson says that \( E \) admits a flat connection (see [13, p. 40, Corollary 3.10]). On the other hand, \( X \) is simply connected because it is rationally connected ([4, p. 545, Theorem 3.5], [7, p. 362, Proposition 2.3]). Therefore, any flat vector bundle on \( X \) is trivial. In particular, the vector bundle \( E \) is trivial. \( \square \)

As before, let \( E \) be a vector bundle over the rationally connected variety \( X \). Let \( r \) be the rank of \( E \).

**Theorem 2.2.** Assume that for every morphism

\[
\gamma : \mathbb{CP}^1 \rightarrow X,
\]

there is a line bundle \( L(\gamma) \rightarrow \mathbb{CP}^1 \) such that \( \gamma^*E = (\gamma^*E)^{\oplus r} \). Then there is a line bundle \( \xi \rightarrow X \) such that \( E = \xi^{\oplus r} \).
Proof. The above condition on $\gamma^* E$ and Proposition 2.1 ensure that the vector bundle $\text{End}(E)$ is trivial. This implies that, for any $x_0 \in X(\mathbb{C})$, the evaluation map

$$H^0(X, \text{End}(E)) \longrightarrow \text{End}_\mathbb{C}(E(x_0))$$

(6)

is an isomorphism; let $A : E \rightarrow E$ be an isomorphism such that all the eigenvalues $\lambda_1, \ldots, \lambda_r$ of $A(x_0)$ are distinct. As the eigenvalues of $A(x)$ are independent of $x \in X$, it follows that $E$ is isomorphic to the direct sum of the line subbundles

$$L_i := \ker(\lambda_i - A) \subseteq E, \quad 1 \leq i \leq r.$$

Since the evaluation map in (6) is an isomorphism, we have

$$\dim H^0(X, L_i \otimes L_j^*) \leq 1$$

for all $i, j \in [1, r]$. Note that if $H^0(X, L_i \otimes L_j^*) = 0$ for some $i, j$, then

$$\dim H^0(X, \text{End}(E)) < r^2,$$

which contradicts the fact that $\text{End}(E)$ is trivial. For $s_{ij} \in H^0(X, L_i \otimes L_j^*) \setminus \{0\}, i, j \in [1, r]$, the composition $s_{ij} \circ s_{ji}$ is an automorphism of $L_i$, hence each $s_{ij}$ is an isomorphism. This completes the proof of the theorem. \(\square\)

A theorem due to Grothendieck says that any vector bundle over $\mathbb{C}P^1$ decomposes into a direct sum of line bundles [6, p. 126, Théorème 2.1]. Therefore, Theorem 2.2 has the following corollary:

**Corollary 2.3.** If for every morphism $\gamma : \mathbb{C}P^1 \rightarrow X$, the vector bundle $\gamma^* E$ is semistable, then there is a line bundle $\zeta \rightarrow X$ such that $E = \zeta^{\oplus r}$.

Acknowledgements

We thank N. Fakhruddin for useful comments. We thank Carolina Araujo for pointing out [1].

References