Numerical Analysis

A posteriori error estimate for a one-dimensional pollution problem in porous media✩

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Abstract

We are interested in the discretization of a time-dependent pollution problem modeling the mass transfer of contaminant in porous media, by the implicit Euler scheme in time and vertex-centered finite volumes in space. The error estimator consists of two types of computable error indicators, the first one being linked to the time discretization and the second one to the space discretization.

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Résumé

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Abstract

Dans cette Note, nous nous intéressons à la discrétisation d’un problème parabolique intégro-différentiel (P) modélisant le transfert de masse d’un polluant en milieu poreux [2,14], par un schéma d’Euler implicite en temps et par un schéma volumes finis centré sur les nœuds en espace (P) nh. Nous proposons une estimation d’erreur a posteriori de type résidu. Celle-ci est composée de deux indicateurs, l’un lié à la discrétisation temporelle (5), l’autre lié à la discrétisation spatiale (12). Le terme intégral est approché par une formule de quadrature, supposée d’ordre assez élevé, de telle sorte que l’erreur d’intégration numérique soit négligée par rapport à l’erreur d’approximation.

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Nous démontrons la borne supérieure globale et la borne inférieure locale pour chaque indicateur d’erreur (voir section 3).

1. Introduction

The finite volumes method (FVM) is a well-adapted method for the discretization of various types of partial differential equations and is extensively used in several engineering fields, see [10]. The mathematical analysis of the method has started only recently. In the context of finite element methods (FEM), an impressive amount of works has been done concerning the theory of a posteriori estimates and mesh adaptivity for a large class of equations; see, for instance [16]. Whereas, for FVM, the situation has not yet been as thoroughly explored, up to now only a few such results had been obtained, see for instance [5,12,19] and references therein. Plenty of work has been done concerning the a posteriori analysis of parabolic type problems. Part of it (see for instance [9,11]) deals only with time scheme adaptivity. However, the authors in [8] deal only with the space discretization and provide appropriate error indicators for it. Another concept consists in establishing a full time and space variational formulation of the continuous problem and using a discontinuous Galerkin method for the discretization with respect to all variables, see for instance [18]. In this Note, we follow a different approach, according to an idea of [7], which consists in introducing two different methods has started only recently. In the context of finite element methods (FEM), an impressive amount of works has been done concerning the theory of a posteriori estimates and mesh adaptivity for a large class of equations; see, for instance [16]. Whereas, for FVM, the situation has not yet been as thoroughly explored, up to now only a few such results had been obtained, see for instance [5,12,19] and references therein. Plenty of work has been done concerning the a posteriori analysis of parabolic type problems. Part of it (see for instance [9,11]) deals only with time scheme adaptivity. However, the authors in [8] deal only with the space discretization and provide appropriate error indicators for it. Another concept consists in establishing a full time and space variational formulation of the continuous problem and using a discontinuous Galerkin method for the discretization with respect to all variables, see for instance [18]. In this Note, we follow a different approach, according to an idea of [7], which consists in introducing two different types of error indicators, both of them of residual type, one for the time discretization and one for the space discretization, and uncoupling as far as possible, the estimates of the time and space errors. It must be noted that both kinds of indicators only depend on the fully discrete solution and the data, so that computing them is easy and not expensive.

In this paper we give a posteriori error estimates for the following integro-differential parabolic problem. For $T > 0$, and $g, f \in L^2(I)$, and $u_0 \in H^1(I)$: $u_0(1) = g(0)$ given functions, find $u$ verifying:

\[
(P) \begin{cases}
\partial_t u + Au = f + v \int_0^t u(s) \, ds & \text{in } Q_T = (-1, 1) \times (0, T), \\
\partial_x u(-1, t) = 0, u(1, t) = g(t) & \text{in } (0, T), \\
u(x, 0) = u_0(x) & \text{in } I = (-1, 1),
\end{cases}
\]

where $Au = -D \frac{\partial^2 u}{\partial x^2} + \frac{\partial (vu)}{\partial x} + (1 - v)u$, $v \in C^2(I)$ is the transport velocity, $D > 0$ is the diffusion coefficient and $v$ is nonnegative constant. For the formulation of this problem see [4,6,14]. Let $\delta$ be the function: $\delta(x) = \frac{1}{2} \int_x^{x+1} v(s) \, ds$. One sets, $L^2(I, \delta) = \{ \phi : I \to \mathbb{R} \text{ measurable}, \int_I \phi(x) e^{-\delta(x)} \, dx < +\infty \}$, $(\phi, \psi) = \int_I \phi(x) \psi(x) e^{-\delta(x)} \, dx$, $\forall (\phi, \psi) \in (L^2(I, \delta))^2$ and $\| \phi \|^2_{L^2(I, \delta)} = (\phi, \phi)$ its associated norm. We need spaces $H^1_{\Gamma_D}(I) = \{ \phi \in H^1(I); \phi(1) = 0 \};$ $W(\text{resp. } W_0) = \{ \phi \in L^2(0, T; H^1_{\Gamma_D}(I)); \phi(0) = 0 \}$ (resp. $L^2(0, T; H^1_{\Gamma_D}(I)))$; $\frac{d}{dt} \phi \in L^2(0, T; L^2(I))$. One defines the function $\theta$ of class $C^1(I)$ by: $\theta(x) = (1 - v + \frac{d}{dt} v(x))$, $\forall x \in I$, and $a(\phi, \omega) = D(\frac{d}{dt} a(\phi, \omega) + (\theta \phi, \omega)$, $\forall (\phi, \omega) \in H^1(I) \times H^1(I)$.

**Definition 1.1.** A function $u \in W$ is a weak solution of problem $(P)$ if $u - g \in W_0; u(0) = u_0$ in $L^2(I)$ and verifies for all $t \in (0, T)$ the variational equation:

\[
\left(\frac{\partial}{\partial t} u(t) \phi \right) + a(u(t), \phi) = (f \phi) + v \left( \int_0^t u(s) \, ds \phi \right), \quad \forall \phi \in H^1_{\Gamma_D}(I).
\]

The existence and uniqueness of solution of problem $(P)$ is proved in [6]. We assume that exists a nonnegative constant $\beta > 0$ s.t. $\theta(x) \geq \beta$, $\forall x \in I$, which rules out all dominant advection problems.

2. A finite volume discretization

In the present section we shall discretize the considered problem $(P)$ by implicit Euler scheme in time and vertex-centered finite volume method in space. In what follows, the symbol $\| \cdot \|$ (resp. $\| \cdot \|_V, V \subset I$) denotes the norm
only depending on a condition, that is, $k > I$ and approximation of the exact solution of the internal nodes, that is, $I$. Let us now consider the following finite volumes approximation of the set $K$ of the dual decomposition $V_h^n$. We may construct another partition of $I$, denoted by $\mathcal{Q}_h^n$ and formed by intervals $Q$ defined by $Q = [V_i \cap T]$. We also need to define the set $K_h^n = \bigcup K$, formed by the intervals $K$ having $x_{V_i}$ (the center of the control volume $V_i$) as a bound and $\gamma$ as the second one. Next we define the spaces $\mathcal{P}_1(T_h^n) := \{v_h \in C^0(I); v_h|T \in \mathcal{P}_1; \forall T \in T_h^n\}$, and $\mathcal{P}_0(V_h^n) := \{v_h \in L^2(I); v_h|V_i \in \mathcal{P}_0; i = 1, \ldots, M\}$, where $\mathcal{P}_1$ is the set of polynomial functions of degree $\leq l$. We denote by $h_T$ (resp. $h_i^n$) the length of $T \in T_h^n$ (resp. $V_i \in V_h^n$) and, for a node $z$ which is shared between $T_1$ and $T_2$, $h_z$ is the minimum of $h_{T_1}$ and $h_{T_2}$. Setting $V_{h,0} = V_h \cap H_{T_1}(I) = \{v_h \in \mathcal{P}_1(T_h^n) \text{ and } v_h(1) = 0\}$, and denoting by $u_{h,0}^n$ the approximation of the exact solution $u^n$ in $V_{h,0}$, namely $u_{h,0}^n = \sum_{i=1}^{M-1} u_i^n \psi_i(x)$ with $\psi_i$ is the shape function associated with node $x_i$. Let us now consider the following finite volumes approximation of $(P)$:

$$P_h^n = \begin{cases} \text{Find } (u_{h,i}^n - g_{h,i}^n)_{0 \leq n \leq N} \in (V_h,0)^{N+1} \text{ such that: } & u_{h,0}^n = \Pi_h u_0^n \text{ in } \Omega, \\ \qquad \qquad \text{and } & \int_{V_i} u_{h,i}^n - u_{\theta h,i}^n - \frac{\tau_n}{\beta} \|f\|^2 \ dx + \int_{V_i} \theta_h u_{h,i}^n \ dx = \int_{V_i} f \ dx + \nu \int_{V_i} \mathcal{I} u_{h,i}^n(x) \ dx, \end{cases}$$

for all $i = 1, \ldots, M, n = 1, \ldots, N$,

where $\Pi_h$ is the $L^2$-projection on $V_{h,0}$.

The numerical diffusion flux function $F_{h,i+\frac{1}{2}}^n$ is chosen such that we have the local conservativity. In order to calculate this numerical diffusion flux, we define the primal partition of $I$ by: $(T_h^n)_{h>0} : \{x_i\}_{i=1,...,M}$ with $I_i = [x_{i-1}, x_i] \forall i = 1, \ldots, M - 1$ and $h_{i+\frac{1}{2}} = x_{i+1} - x_i$, and the corresponding dual partition by: $(\mathcal{V}_h^n)_{h>0}$ with $V_i = [x_{i-1}, x_{i+1}] \forall i = 2, \ldots, M - 1, V_1 = [x_1, x_\frac{1}{2}]$ with $x_1 = x_\frac{1}{2}$ and $V_M = [x_{M-\frac{1}{2}}, x_M]$ with $x_M = x_{M+\frac{1}{2}}$. Therefore, $F_{h,i+\frac{1}{2}}^n = \frac{\partial u_{h,i}^n}{\partial x}(x_{i+\frac{1}{2}})$, which yields $F_{h,i+\frac{1}{2}}^n = \frac{u_{h,i+1}^n - u_{h,i}^n}{h_{i+\frac{1}{2}}}$. The approximation of $v$ (resp. $\theta$, $v_h$ (resp. $\theta_h$), is a piecewise polynomial of degree smaller than a fixed integer $l$ and such that there exists a constant $c(v)$ (resp. $c(\theta)$) only depending on $v$ (resp. $\theta$) satisfying

$$\|v - v_h\|_{L^\infty(I)} \leq c(v) h^{l+1} \quad \text{and} \quad \|\theta - \theta_h\|_{L^\infty(I)} \leq c(\theta) h^{l+1}.$$

The quantity $\mathcal{I} u_{h,i}^n$ is a numerical approximation of the integral $\int u^n$ and there exist two constants $c(u)$ depending on $u$ and $k > 1$ a fixed constant such that

$$\|\mathcal{I} u_{h,i}^n - \int u^n\|_V \leq c(u) \tau_k h V.$$
Remark 1. The terms $\int_{V_i} \frac{u^n_h - u^{n-1}_h}{\tau_n} \ dx$, $\int_{V_i} v_h \frac{\partial u^n_h}{\partial x} \ dx$ and $\int_{\Omega} \theta_h u^n_h \ dx$ can be approximated by $h_i \frac{u^n_h - u^{n-1}_h}{\tau_n}$, $v_i [u^n_h(x_i) - u^n_h(x_{i-\frac{1}{2}})] = v_i [u^n_i - u^n_{i-1}]$ for upwind scheme and $h_i \theta_i u^n_i$ respectively where $u^n_i = u^n_h(x_i)$, $v_i = v_h(x_i)$ and $\theta_i = \theta_h(x_i)$, the existence and uniqueness of a solution of this scheme is proved in [1].

3. A posteriori error analysis

3.1. A posteriori error analysis of the time discretization

By analogy to [7,11,13], for each $n, 1 \leq n \leq N$, we define the time error indicators

$$\eta^u_n = \left( \frac{\tau_n}{3} \right)^{\frac{1}{2}} \left( D^\frac{1}{2} \| \partial_t \left( u^n_h - u^{n-1}_h \right) \| + \beta \| \| u^n_h - u^{n-1}_h \|. \right) \ . \quad (5)$$

It can be observed that, once the discrete solution $(u^n_h)_{0 \leq n \leq N}$ is known, the previous error indicators are very easy to compute.

Proposition 3.1. The following time upper error estimate holds, for $1 \leq n \leq N$,

$$\| u - u_\tau \|_{(t_n)} \leq \left( 1 + \sigma^2_\tau \right) \| u_\tau - u_{\tau h} \|_{(t_n)} + \left( \sum_{m=1}^{n} (\eta^u_m)^2 \right)^{\frac{1}{2}} \quad (6)$$

Moreover, we have the following time lower error bound

$$\eta^u_n \leq \left( \frac{\| \partial_t (u - u_\tau) \|_{L^2(t_{n-1},t_n;L^2(I,\delta))} + D^\frac{1}{2} \| \partial_t (u - u_\tau) \|_{L^2(t_{n-1},t_n;L^2(I,\delta))} + \| u - u_\tau \|_{L^2(t_{n-1},t_n;L^2(I,\delta))} \right) + c_n(u_0, f) \tau_n + \| a^n - a^n_h \| + \sigma^2_\tau \| u^{n-1} - u^{n-1}_h \| \ . \quad (7)$$

where the constant $c_n(u_0, f)$ is given by $c_n(u_0, f) = \frac{\nu}{\sqrt{2}} \beta^{-\frac{1}{2}} t_n^\frac{1}{2} K^\frac{1}{2} e^{LT}$.

Proof. For $\phi \in H^1_{I_{\partial}^+}(I)$, we have

$$\left( \frac{\partial (u - u_\tau)}{\partial t}, \phi \right) + a(u - u_\tau, \phi) = -a(u_\tau - u^n, \phi) + v \left( \int_{t_n}^t u(s) \ ds / \phi \right) \ . \quad (8)$$

taking $\phi = u - u_\tau$, integrating over $[t_{n-1}, t_n]$, sum up on the $n$, and applying the Hölder’s inequality, and owing to the definitions of $u_\tau$, $\sigma_\tau$ and $\eta^u_m$ we get the time upper error bound. By virtue of (5), taking $\phi = a^n - a^{n-1}$ in (8), integrating over $(t_{n-1}, t_n)$, and applying the Hölder’s inequality, this yields the time lower error bound. \square

3.2. A posteriori error analysis of the space discretization

We now introduce the usual nodal basis $\{ \phi_z \}_{z \in Z_h}$ for $V_h$, whence $\{ \phi_z \}_{z \in Z_h}$ is the nodal basis for $V_{h,0}$, and the following quasi-interpolants given in [15], defined for a given $\psi \in H^1(I)$ by: $I_h \psi \in V_h$ as $I_h \psi := \sum_{z \in Z_h} \frac{h_z}{h_z} \phi_z \psi \phi_z$, and $I_{h,0} \psi \in V_{h,0}$ as $I_{h,0} \psi := \sum_{z \in Z_h} \frac{h_z}{h_z} \phi_z \psi \phi_z$. Let $T \in T_h^\circ$ and let $\omega_T$ be the union of $T$ and the one or two adjacent elements. Let $z \in Z_h$ and let $V$ be a control volume which $z$ belongs to, using [17, Lemma 3.1], we derive

$$\sum_{z \in Z_h \cap \omega_T} h_z^{-1} (I_{h,0} \psi)^2 (z) \leq |\psi|_{L^1,av}^2 \ . \quad (9)$$
Lemma 3.2. Let ψ be an element of $H^{1}_{0}(I)$ and $\psi_{h} = I_{h,0}\psi$, and let $\overline{\psi_{h}}$ be the mean value of $\psi_{h}$ over the control volume $V$, then one has the following estimates:

$$\sum_{z \in Z_{h} \cap V} H_{z}^{-1}(\psi_{h} - \overline{\psi_{h}})^{2}(z) \leq |\psi|_{1,\omega V}^{2}, \quad \|\psi_{h} - \overline{\psi_{h}}\|_{0,V} \leq h_{V} |\psi|_{1,\omega V}, \quad (10)$$

and

$$\|\psi_{h} - \overline{\psi_{h}}\|_{0,V} \leq h_{V} |\psi|_{1,\omega V}. \quad (11)$$

In order to obtain an a posteriori estimate in each control volume $V_{i}$, let us define the following local spatial indicators

$$\eta_{n,V_{i}}^{2} := h_{i}^{2} \|R_{n}^{0}\|_{V_{i}}^{2} + \sum_{z \in Z_{h} \cap V_{i}} h_{i}(D \left[ \frac{\partial u_{n}^{i}}{\partial x} \right](z))^{2} + \sum_{z \in Z_{h} \cap V_{i}} h_{i}(D \left[ \frac{\partial u_{n}^{i}}{\partial x} \right](z))^{2}, \quad (12)$$

where $R_{n}^{0} = f_{n} + v\int u_{n}^{i} - \frac{u_{n}^{i} - u_{n}^{i-1}}{\tau_{n}} + D \frac{\partial u_{n}^{i}}{\partial x} - v \frac{\partial u_{n}^{i}}{\partial x} - \partial u_{n}^{i}$. We have the following upper bounds for the error:

Proposition 3.3. If $u$ is the solution of problem (1) and $u_{h,\tau}$ is a continuous approximation, linear on time and on space defined from the value $(u_{n})_{n}$, solution of problem $(P)_{n}^{u}$, then we have the following estimate:

$$\|u - u_{h,\tau}\|(t_{n}) \leq \left( \sum_{m=1}^{n} \tau_{m} \sum_{i=1}^{M} \left( u_{m,V_{i}}^{2} + \frac{h_{i}^{2}}{2} f - f_{h} \right)^{2} \right) \frac{1}{2} + \|u_{0} - \Pi_{h} u_{0}\| + c_{n}^{u}(u_{0}, f) h^{(l+1)}$$

$$+ c_{n}(u) h^{k+1} + h^{2} \tau B_{n}(u_{n}^{0}), \quad (13)$$

where $c_{n}^{u}(u_{0}, f) = \|u_{0}\| + \tau_{n}^{\frac{1}{2}} \|f\| + \tau_{n}^{\frac{1}{2}} c^{0}_{0} + \tau_{n}^{\frac{1}{2}} \|u_{0}\|$, $c_{n}(u) = c(u)\tau_{n}^{\frac{1}{2}}$, and

$$B_{n}(u_{n}^{0}) = \left( \sum_{m=1}^{n} \sum_{i=1}^{M} \left( D \frac{\partial u_{n}^{i}}{\partial x} \right)^{2}_{0,\omega V_{i}} + \left( \frac{\partial u_{n}^{i}}{\partial x} \right)^{2}_{0,\omega V_{i}} \right) \frac{1}{2}.$$  

We have also the following lower bounds for the error:

Proposition 3.4. For $f \in L^{2}(I)$ and $u_{0} \in H^{1}(I)$, we have the following local estimate:

$$\eta_{n,V_{i}} \leq h_{i} \left\| \frac{\partial (u_{\tau} - u_{h,\tau})}{\partial t} \right\|_{0,V_{i}} + \left\| D \frac{\partial (u_{n} - u_{h}^{n})}{\partial x} \right\|_{0,V_{i}} + h_{i} \|f - f_{h}\|_{0,V_{i}} + h_{i} \|u^{n} - u_{h}^{n}\|_{0,V_{i}}$$

$$+ h^{l+1} c_{n}(u_{n}^{0}) + h^{2} \tau^{k} c(u), \quad (14)$$

where $c_{n}(u_{n}^{0}) = c(v) \|u_{n}^{0}\|_{0,V_{i}} + c(\theta) \|u_{n}^{0}\|_{0,V_{i}}$.

For the proofs of the Propositions 3.2 and 3.3, we can see [3].

Remark 2. Some constants $c(u)$ appearing in the upper and lower bounds depending on the exact solution $u$, but pondering with coefficient of higher order. This is due to the numerical integration order.

We do not actually show an a posteriori estimator for the multidimensional case or the advection dominated case ($\beta < 0$); this is of interest but lies outside the scope of this Note.

References