The explicit characterization of coefficients of a.e. convergent orthogonal series

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Abstract
We characterize sequences of numbers \((a_n)\) such that \(\sum_{n \geq 1} a_n \Phi_n\) converges a.e. for any orthonormal system \((\Phi_n)\) in any \(L^2\)-space. To cite this article: A. Paszkiewicz, C. R. Acad. Sci. Paris, Ser. I 347 (2009).

Résumé
Caractérisation explicite des coefficients des séries orthogonales convergentes presques partout. On donne une complète caractérisation de la suite des nombres \((a_n)\) telle que \(\sum_{n \geq 1} a_n \Phi_n\) converge, presque partout, pour tout système orthogonal \((\Phi_n)\) dans tout espace \(L^2\).


This Note presents a complete characterization of sequences \((a_n)\) for which:

\[(a) \sum a_n \Phi_n\) converge a.e. for any orthonormal (O.N. for short) sequence \((\Phi_n)\) in any \(L^2\)-space.

The main result stated in Theorem 6 below is proved in [2]. Without loss of generality we consider only sequences \((a_n)\) satisfying \(a_n \geq 0, \sum n \geq 1 a_n^2 \leq 1\). Let such a sequence be fixed and let

\[A = \left\{ \sum a_n^2; m = 1, 2, \ldots \right\}.\]  \hfill (1)

It is well known that a very sharp sufficient condition for \((a)\) can be formulated by the use of so-called majorizing measures. We say, as in [2, Definition 1.7], that \(m\) is a majorizing measure on \(A\) if \(m\) is a Borel measure on \(\mathbb{R}\) concentrated on \(A\), and

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\[
\int_0^1 \frac{d\epsilon}{\sqrt{m((t - \epsilon^2, t + \epsilon^2))}} \leq 1 \quad \text{for any } t \in A.
\]

The existence of a finite majorizing measure on \( A \) implies the a.e.-convergence of \( \sum a_n \Phi_n \) for any O.N.-system \( (\Phi_n) \). This can be obtained from [3, Theorems 4.6 and 2.9], the details are explained in [6] and also in [2, Sections 2.9, 2.10].

Nevertheless, an explicit characterization of coefficients \( a_n, n \geq 1 \), satisfying (a) was an open problem for decades, as indicated by a number of authors (see V.F. Gaposhkin [1], M. Talagrand [4]).

A solution is presented in this Note. We show, in particular, that the existence of a finite majorizing measure on \( A \) is equivalent to condition (a), and we construct a majorizing measure \( m_\bar{A} \) on the closure \( \bar{A} \) with the smallest total mass \( m_\bar{A}(\bar{A}) \).

Our new formulas giving explicit characterizations of sequences \( (a_n) \) satisfying (a), are complicated. It is interesting to present them together with a simpler characterization of unconditional a.e.-convergence of series \( \sum a_n \Phi_n \), announced in [2].

**Definition 1.** Let us denote \( d^k_n = [\frac{n}{2^k}, \frac{n+1}{2^k}) \), \( 0 \leq n < 2^k \), for \( k \geq 1 \), and let

\[
\Delta^A_k = \bigcup_{n \in \Sigma_k} d^k_n
\]

with

\[
\Sigma_k = \{ n = 0, \ldots, 2^k - 1; \; d^k_n \cap A \neq \emptyset \}, \quad k \geq 1.
\]

By \( \| \cdot \| \) we denote the \( L_2 \)-norm in \( L_2[0, 1) \) or in another \( L_2 \)-space of real functions, writing \( \| h \| = \infty \) when \( \int |h|^2 = \infty \). As usual, \( 1_Z(\cdot) \) is an indicator of the set \( Z \).

Relatively simple characterizations can be formulated for a.e.-convergence of permutations of the series \( \sum a_n \Phi_n \) in (a) as follows:

**Theorem 2.** (See [2, Theorem 1.2].) The following conditions are equivalent:

(b) there exists a permutation \( \sigma \) on the set \( \mathbb{N} \) of positive integers such that

\[
\sum_{n \geq 1} a_{\sigma(n)} \Phi_n \quad \text{converges a.e. for any O.N.-system } (\Phi_n);
\]

(2)

\[
(\beta) \quad \| \sum_{k \geq 1} 1_{\Delta^A_k} \| < \infty \quad \text{for } A \text{ given by (1)}.
\]

**Theorem 3.** (See [2, Theorem 1.3].) The following conditions are equivalent:

(c) for any permutation \( \sigma \) of \( \mathbb{N} \), (2) is satisfied;

(\gamma) \quad \sum_{k \geq 1} \| 1_{\Delta^A_k} \| < \infty \quad \text{for } A \text{ given by (1)}.

Obviously,

\[
(c) \implies (a) \implies (b),
\]

and thus any condition (\( \alpha \)) equivalent to (a) should satisfy

\[
(\gamma) \implies (\alpha) \implies (\beta).
\]

It turns out that (a) can be obtained by the following more delicate analysis of the indicators \( 1_{\Delta^A_k} \):

**Definition 4.** For any \( k \geq 1 \), let \( \mathcal{F}_k = \sigma(d^k_n; 0 \leq n < 2^k) \) be the \( \sigma \)-field generated by the intervals \( d^k_n = [\frac{n}{2^k}, \frac{n+1}{2^k}) \). By \( \| h \|_k \) we denote the ‘conditional \( L_2 \)-norm’

\[
\| h \|_k = \left( \mathbb{E}(h^2 | \mathcal{F}_k) \right)^{\frac{1}{2}}
\]
for a real $L_2$-function $h$ on $[0, 1)$, where $E(\cdot | \mathcal{F}_k)$ denotes the conditional expectation in $[0, 1)$ with respect to Lebesgue measure $\lambda$. Thus $\|h\|_k$ is $\mathcal{F}_k$-measurable.

**Definition 5.** For $L_2$-functions $h : [0, 1) \to [0, \infty)$ we define (non-linear) operations

$$V_A^k h = 1_{\Delta^k_I} + \|h\|_k, \quad k \geq 1.$$  

The main result can be formulated in the following way:

**Theorem 6.** (See [2, Theorem 1.8].) For a sequence of coefficients $(a_n)$, $\sum a_n^2 \leq 1$, the following conditions are equivalent:

(a) $\sum_{n \geq 1} a_n \Phi_n$ converges a.e. for any O.N. sequence $(\Phi_n)$;

(A) there exists a majorizing measure $m$ on $A$ with $m(A) < \infty$ for $A$ given by (1);

(α) $\lim_{l \to \infty} \|V_1^A \cdots V_l^A 0\| < \infty$.

If conditions (a), (A) or (α) are not satisfied, then $\sum_{n \geq 1} a_n \Phi_n$ diverges a.e. for some O.N. sequence $(\Phi_n)$.

If conditions (a), (A) or (α) are satisfied, then we can construct some canonical majorizing measure $m_{\bar{A}}$ on $\bar{A} = A \cup \{0\}$ with minimal total mass $m_{\bar{A}}(\bar{A})$. To do this we introduce the following operations:

**Definition 7.** For an $L_2$-function $h : [0, 1) \to [0, \infty)$ we define

$$W_k^A h = \frac{\|h\|_k + 1}{\|h\|_k} h$$

with the convention $a^0_0 = 0$ for $a \geq 0$. Let $m_l^A$ be the measure on $[0, 1)$ with density $dm_l^A/d\lambda = (W_1 \cdots W_{l-1} 1_{\Delta^l_I})^2$, $l \geq 1$, for $\Delta^l_I$ given by Definition 1.

**Theorem 8.** (See [2, Theorem 8.11].) The measures $m_l^A$ converge weakly, for $l \to \infty$, to some measure $m_{\bar{A}}$ concentrated on the closure $\bar{A}$ and $2m_{\bar{A}}$ is a majorizing measure on $\bar{A}$ with

$$2m_{\bar{A}}(\bar{A}) \leq C \inf\{m(A); \ m - a majorizing measure on A\},$$

for some constant $C$.

Moreover, any majorizing measure $m_{\bar{A}}$ on $\bar{A}$ is a weak limit of a sequence of some majorizing measures on $A$ [2, Proposition 1.9].

In fact for $M(A)$ and $N(A)$ being any two of the following three functions

$$A \mapsto \lim_{l \to \infty} \|V_1^A \cdots V_l^A 0\|,$$

$$A \mapsto \lim_{l \to \infty} \|W_1 \cdots W_{l-1} 1_{\Delta^l_I}\| = \sqrt{m_{\bar{A}}(\bar{A})},$$

and

$$A \mapsto \sup_{\Phi_n - O.N.-system} \|\sup_{n \geq 1} |a_1 \Phi_1 + \cdots + a_n \Phi_n|\|,$$

defined for all sets $A$ of the form (1), $M$ and $N$ are of the same ‘size’, i.e.,

$$\frac{1}{C} M(A) - C \leq N(A) \leq CM(A) + C$$

for some universal constant $C$. 
Moreover, K. Tandori has proved (see [5] and [2, Theorems 8.4, 8.4*]) that for any O.N.-system \( (\Phi_n) \) condition (a) is equivalent to
\[
\left\| \sup_{n \geq 1} |a_1 \Phi_1 + \cdots + a_n \Phi_n| \right\| < \infty.
\]

The main difficulty in the proof of Theorem 6 is the construction, for a given finite sequence \( (a_n)_{n \leq N} \), of a system \( (\Phi_n)_{n \leq N} \) such that \( \left\| \sup_{1 \leq n \leq N} |a_1 \Phi_1 + \cdots + a_n \Phi_n| \right\| \) is maximal possible. This is done in [2, Sections 3–7].

References