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COMPTES RENDUS MATHEMATIQUE

Probability Theory

Moment identities for Poisson–Skorohod integrals and application to measure invariance ☆

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Abstract

We present a moment identity on the Poisson space that extends the Skorohod isometry to arbitrary powers of the Skorohod integral. Applications of this identity are given to the invariance of Poisson measures under intensity preserving random transformations. ***To cite this article: N. Privault, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Identités de moments pour les intégrales de Poisson–Skorohod et applications à l'invariance en mesure. Nous présentons une identité de moments sur l'espace de Poisson qui étend l'isométrie de Skorohod à des puissances quelconques de l'intégrale de Skorohod, et nous étudions les applications de cette identité à l'invariance de la mesure de Poisson sous les tranformations aléatoires qui préservent l'intensité. ***Pour citer cet article : N. Privault, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

The classical invariance theorem for Poisson measures states that given a *deterministic* transformation $\tau : X \to Y$ between measure spaces (X, σ) and (Y, μ) sending σ to μ, the corresponding transformation on point processes maps the Poisson distribution π_σ with intensity $\sigma(\mathrm{d}x)$ on X to the Poisson distribution π_μ with intensity $\mu(\mathrm{d}y)$ on Y. In this Note we present sufficient conditions for the invariance of *random* transformations $\tau : \Omega^X \times X \to Y$ of Poisson random measures on metric spaces. Our results are inspired by the treatment of the Wiener case in [8], see [6] for a recent simplified proof. However, the use of *finite difference* operators instead of *derivation* operators as in the continuous case makes the proofs and arguments more complex from an algebraic point of view. Here the almost sure isometry condition on $\mathbb{R}^d$ assumed in the Gaussian case will be replaced by an almost sure condition on the preservation of intensity measures and, as in the Wiener case, we will characterize probability measures via their moments.

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In this Note the proofs of the main results are only outlined. The details of the complete proofs, which are technical, can be found in [5].

2. Notation and preliminaries

In this section we recall some notation and facts on stochastic analysis under Poisson measures, see [3] and [7] for recent reviews. Let X be a σ-compact metric space with Borel σ-algebra $\mathcal{B}(X)$. Let Ω^X denote the configuration space on X, i.e. the space of at most countable and locally finite subsets of X, defined as

$$\Omega^X = \big\{\omega = (x_i)_{i=1}^N \subset X,\ x_i \neq x_j\ \forall i \neq j,\ N \in \mathbb{N} \cup \{\infty\}\big\},$$

and endowed with the Poisson probability measure π_σ with σ-finite diffuse intensity $\sigma(\mathrm{d}x)$ on X. Each element ω of Ω^X is identified to the Radon point measure $\omega = \sum_{i=1}^{\omega(X)} \epsilon_{x_i}$, where ϵ_x denotes the Dirac measure at $x \in X$ and $\omega(X) \in \mathbb{N} \cup \{\infty\}$ is the cardinality of ω. Let D denote the finite difference gradient defined $\pi_\sigma \otimes \sigma(\mathrm{d}\omega, \mathrm{d}x)$-almost everywhere as

$$D_x F(\omega) = F\big(\omega \cup \{x\}\big) - F(\omega), \quad \omega \in \Omega^X,\ x \in X, \tag{1}$$

for any random variable $F : \Omega^X \to \mathbb{R}$, cf. [1]. We refer to [2] for the definition of the Skorohod integral operator

$$\delta_\sigma(u) = \int_X u\big(\omega \setminus \{t\}, t\big)\big(\omega(\mathrm{d}t) - \sigma(\mathrm{d}t)\big), \tag{2}$$

on any sufficiently integrable measurable process $u : \Omega^X \times X \to \mathbb{R}$. Note that if $D_t u_t = 0$, $t \in X$, $\delta_\sigma(u)$ coincides with the compensated Poisson–Stieltjes integral of u. From Corollary 1 in [4] we have the duality relation

$$E_\sigma\big[\langle DF, u\rangle_{L^2(X,\sigma)}\big] = E_\sigma\big[F\delta_\sigma(u)\big], \qquad F \in \mathrm{Dom}(D),\ u \in \mathrm{Dom}(\delta_\sigma). \tag{3}$$

In addition, for any $u \in \mathrm{Dom}(\delta_\sigma)$ we have the commutation relation

$$D_t \delta_\sigma(u) = \delta_\sigma(D_t u) + u_t, \quad t \in X. \tag{4}$$

3. Moment identities

Using relations (3) and (4), the next lemma provides an extension of the Skorohod isometry to moments of order higher than 2. Here and in other formulas stated in the sequel we will simply assume that all terms are sufficiently summable and integrable.

Lemma 1. *We have*

$$\begin{aligned} E_\sigma\big[\big(\delta_\sigma(u)\big)^{n+1}\big] = {} & \sum_{k=0}^{n-1} \binom{n}{k} E_\sigma\bigg[\int_X (u_t)^{n-k+1}\big(\delta_\sigma(u)\big)^k \sigma(\mathrm{d}t)\bigg] \\ & + \sum_{k=1}^{n} \binom{n}{k} E_\sigma\bigg[\int_X (u_t)^{n-k+1}\Big(\big(\delta_\sigma\big((I + D_t)u\big)\big)^k - \big(\delta_\sigma(u)\big)^k\Big)\sigma(\mathrm{d}t)\bigg], \end{aligned}$$

for all $n \geqslant 1$.

Proof. This lemma is proved using the identities (3) and (4) applied to $F = (\delta_\sigma(u))^n$. □

Lemma 1 also shows that the moments of the compensated Poisson stochastic integral $\int_X f(t)(\omega(\mathrm{d}t) - \sigma(\mathrm{d}t))$ of $f \in \bigcap_{p=1}^{N+1} L_\sigma^p(X)$ satisfy the recurrence identity

$$E_\sigma\bigg[\bigg(\int_X f(t)\big(\omega(\mathrm{d}t) - \sigma(\mathrm{d}t)\big)\bigg)^{n+1}\bigg] = \sum_{k=0}^{n-1} \binom{n}{k} \int_X \big(f(t)\big)^{n-k+1} \sigma(\mathrm{d}t) E_\sigma\bigg[\bigg(\int_X f(t)\big(\omega(\mathrm{d}t) - \sigma(\mathrm{d}t)\big)\bigg)^k\bigg]. \tag{5}$$

In particular, in order for $\delta_\sigma(u)$ to have the same moments as the compensated Poisson integral of f, it should satisfy the recurrence relation

$$E_\sigma\big[\big(\delta_\sigma(u)\big)^{n+1}\big]=\sum_{k=0}^{n-1}\binom{n}{k}\int_X\big(f(t)\big)^{n-k+1}\sigma(\mathrm{d}t)E_\sigma\big[\big(\delta_\sigma(u)\big)^k\big], \tag{6}$$

$n\geqslant 0$, which is relation (5) for the moments of compensated Poisson stochastic integrals, and characterizes their distribution by Carleman's condition when $\sup_{p\geqslant 1}\|f\|_{L^p_\sigma(Y)}<\infty$. In order to simplify the presentation of moment identities for the Skorohod integral δ_σ, it will be convenient to use the symbolic notation

$$\Delta_{s_0}\cdots\Delta_{s_j}\prod_{p=0}^{j}u_{s_p}:=\sum_{\substack{\Theta_0\cup\cdots\cup\Theta_j=\{0,1,\dots,j\}\\ 0\notin\Theta_0,\dots,j\notin\Theta_j}}D_{\Theta_0}u_{s_0}\cdots D_{\Theta_j}u_{s_j}, \tag{7}$$

where $D_\Theta=\prod_{j\in\Theta}D_{s_j}$ when $\Theta\subset\{0,1,\dots,j\}$, $j\geqslant 0$, $s_0,\dots,s_j\in X$.

Let (Y,μ) denote another measure space with associated configuration space Ω^Y and Poisson measure π_μ with intensity $\mu(\mathrm{d}y)$.

Theorem 1. *Let $N\geqslant 0$ and let $R:L^p_\mu(Y)\to L^p_\sigma(X)$ be a random isometry for all $p=2,\dots,N+1$. Then for $h\in\bigcap_{p=2}^{N+1}L^p_\mu(Y)$ and $n=0,\dots,N$ we have*

$$\begin{aligned}E_\sigma\big[\big(\delta_\sigma(Rh)\big)^{n+1}\big]&=\sum_{k=0}^{n-1}\binom{n}{k}\int_Y\big(h(y)\big)^{n-k+1}\mu(\mathrm{d}y)E_\sigma\big[\big(\delta_\sigma(Rh)\big)^k\big]\\&\quad+\sum_{a=0}^{n}\sum_{j=0}^{a}\binom{a}{j}\sum_{b=a}^{n}\sum_{\substack{l_0+\cdots+l_a=n+1-b\\ l_0,\dots,l_a\geqslant 1\\ l_{a+1},\dots,l_b=1}}C(l_0,\dots,l_a,a,b,n)\\&\quad\left(\prod_{q=j+1}^{b}\int_Y\big(h(y)\big)^{l_q}\mu(\mathrm{d}y)\right)E_\sigma\left[\int_{X^{b+1}}\Delta_{t_0}\cdots\Delta_{t_j}\left(\prod_{p=0}^{j}\big(Rh(t_p)\big)^{l_p}\right)\sigma(\mathrm{d}t_0)\cdots\sigma(\mathrm{d}t_j)\right],\end{aligned}$$

where

$$C(l_0,\dots,l_a,a,b,n)=(-1)^b\binom{n}{l_0-1}\sum_{0=r_{b+1}<\cdots<r_0=a+b}\prod_{q=0}^{b}\prod_{p=q+1-b+r_{q+1}}^{q-b+r_q-1}\binom{l_1+\cdots+l_{p+1}-q-1}{l_1+\cdots+l_p-q}.$$

Proof. This result is obtained by repeated applications of the integration by parts formula (3) until removal of all terms in δ_σ. □

As a consequence of Theorem 1, if $R:L^p_\mu(Y)\to L^p_\sigma(X)$ is a random isometry for all $p=1,\dots,N+1$, that satisfies the condition

$$\int_{X^{j+1}}\Delta_{t_0}\cdots\Delta_{t_j}\left(\prod_{p=0}^{j}\big(Rh(t_p)\big)^{l_p}\right)\sigma(\mathrm{d}t_0)\cdots\sigma(\mathrm{d}t_j)=0, \tag{8}$$

for all $l_0+\cdots+l_j\leqslant N+1$, $l_0\geqslant 1,\dots,l_j\geqslant 1$, $j=1,\dots,N$, then we have

$$E_\sigma\big[\big(\delta_\sigma(Rh)\big)^{n+1}\big]=\sum_{k=0}^{n-1}\binom{n}{k}\int_Y\big(h(y)\big)^{n-k+1}\mu(\mathrm{d}y)E_\sigma\big[\big(\delta_\sigma(Rh)\big)^k\big], \tag{9}$$

$n=0,\dots,N$, i.e. the moments of $\delta_\sigma(Rh)$ satisfy the recurrence relation (5).

Corollary 1. *Let $R : L^p_\mu(Y) \to L^p_\sigma(X)$ be a random isometry for all $p \geqslant 1$, and assume that $h \in \bigcap_{p=1}^\infty L^p_\mu(Y)$ satisfies $\sup_{p\geqslant 1} \|h\|_{L^p_\mu(Y)} < \infty$ and the cyclic condition*

$$D_{t_1} Rh(t_2) \cdots D_{t_k} Rh(t_1) = 0, \quad t_1, \ldots, t_k \in X, \tag{10}$$

$\pi_\sigma \otimes \sigma^{\otimes k}$-a.e., for all $k \geqslant 1$. Then, under π_σ, $\delta_\sigma(Rh)$ has same distribution as the compensated Poisson integral $\delta_\mu(h)$ of h under π_μ.

Proof. We first show that (10) implies (8), and then apply Theorem 1. □

4. Invariance of Poisson measures

Given a measurable random process $\tau : \Omega^X \times X \to Y$, indexed by X, let $\tau_*(\omega)$, $\omega \in \Omega^X$, denote the image measure of ω by τ, i.e.

$$\tau_* : \Omega^X \to \Omega^Y \tag{11}$$

maps $\omega = \sum_{i=1}^{\omega(X)} \epsilon_{x_i} \in \Omega^X$ to $\tau_*(\omega) = \sum_{i=1}^{\omega(X)} \epsilon_{\tau(x_i)} \in \Omega^Y$. In other terms, the random mapping $\tau_* : \Omega^X \to \Omega^Y$ shifts each configuration point $x \in \omega$ according to $x \mapsto \tau(\omega, x)$. We are interested in finding conditions for $\tau_* : \Omega^X \to \Omega^Y$ to map π_σ to π_μ. This question is well known to have an affirmative answer when the transformation $\tau : X \to Y$ is deterministic and maps σ to μ.

Corollary 2. *Let $\tau : \Omega^X \times X \to Y$ be a measure preserving transformation mapping σ to μ, i.e. $\tau_*(\omega, \cdot)\sigma = \mu$, $\omega \in \Omega^x$, and satisfying the cyclic condition*

$$D_{t_1} \tau(\omega, t_2) \cdots D_{t_k} \tau(\omega, t_1) = 0, \quad t_1, \ldots, t_k \in X,\ \omega \in \Omega^X, \tag{12}$$

for all $k \geqslant 1$. Then $\tau_* : \Omega^X \to \Omega^Y$ maps π_σ to π_μ, i.e. $\tau_* \pi_\sigma = \pi_\mu$ is the Poisson measure with intensity $\mu(\mathrm{d}y)$ on Y.

Proof. We apply Corollary 1 to the isometry $R : L^p_\mu(Y) \to L^p_\sigma(X)$, $p \geqslant 1$, defined by $Rh = h \circ \tau$, $h \in L^p_\mu(Y)$. Then we note that (12) implies (10) and that we have $\delta_\sigma(Rh) = \delta_\sigma(h \circ \tau) = \delta_\mu(h) \circ \tau_*$ from (2) and the relation $D_t Rh(t) = D_t h(\tau(\omega, t)) = 0$, $\sigma \otimes \pi_\sigma(\mathrm{d}t, \mathrm{d}\omega)$-a.e. □

In the above corollary the identity (12) is interpreted when Y is a metric space by stating that for all $k \geqslant 1$ and $t_1, \ldots, t_k \in X$ the k-tuples $(\tau(\omega \cup \{t_1\}, t_2), \tau(\omega \cup \{t_2\}, t_3), \ldots, \tau(\omega \cup \{t_{k-1}\}, t_k), \tau(\omega \cup \{t_k\}, t_1))$ and $(\tau(\omega, t_2), \tau(\omega, t_3), \ldots, \tau(\omega, t_k), \tau(\omega, t_1))$ coincide on at least one component in Y^k, for almost every $\omega \in \Omega^X$. Examples of random transformations $\tau : \Omega^X \times X \to Y$ satisfying the above hypotheses are considered in [5].

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