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COMPTES RENDUS MATHEMATIQUE

Complex Analysis

Oka manifolds

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Abstract

We give a positive answer to Gromov's question [Oka's principle for holomorphic sections of elliptic bundles, J. Amer. Math. Soc. 2 (1989) 851–897, 3.4.(D), p. 881]: If every holomorphic map from a compact convex set in a Euclidean space $\mathbb{C}^n$ to a certain complex manifold Y is a uniform limit of entire maps $\mathbb{C}^n \to Y$, then Y enjoys the parametric Oka property. In particular, for any reduced Stein space X the inclusion $\mathcal{O}(X,Y) \hookrightarrow \mathcal{C}(X,Y)$ of the space of holomorphic maps into the space of continuous maps is a weak homotopy equivalence. ***To cite this article: F. Forstnerič, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Les varietes de Oka. Nous donnons une réponse positive à la question suivante posée par Gromov [Oka's principle for holomorphic sections of elliptic bundles, J. Amer. Math. Soc. 2 (1989) 851–897, 3.4.(D), p. 881] : Si une variété analytique complexe Y est telle que toute application holomorphe d'un voisinage d'un sous-ensemble compact convexe de l'espace euclidien $\mathbb{C}^n$ dans Y peut être approximée par des applications entière de $\mathbb{C}^n$ dans Y, alors les applications holomorphes d'un espace de Stein réduit X dans Y possèdent la propriété d'Oka paramétrique. ***Pour citer cet article : F. Forstnerič, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. The Oka–Grauert–Gromov principle

A complex manifold Y is said to enjoy the *Convex Approximation Property* (CAP) if every holomorphic map from a neighborhood of a compact convex set $K \subset \mathbb{C}^n$ to Y can be approximated, uniformly on K, by entire maps $\mathbb{C}^n \to Y$. This property was introduced in [2,1] where it was shown to be equivalent to the basic Oka property of Y.

Here we show that CAP also implies the *parametric Oka property*, thereby providing a positive answer to Gromov's question [7, 3.4.(D), p. 881]. Our main result is the following:

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Theorem 1.1. *If $h : Z \to X$ is a stratified holomorphic fiber bundle over a reduced Stein space X such that all its fibers satisfy CAP, then sections $X \to Z$ satisfy the parametric Oka principle. In particular, the inclusion of the space of all holomorphic sections $X \to Z$ into the space of all continuous sections is a weak homotopy equivalence.*

The conclusion of Theorem 1.1 means the following:

Given a compact $\mathcal{O}(X)$-convex subset K of X, a closed complex subvariety X' of X, compact sets $P_0 \subset P$ in a Euclidean space $\mathbb{R}^m$, and a continuous map $f : P \times X \to Z$ such that (a) *for every $p \in P$, $f(p, \cdot) : X \to Z$ is a section of $Z \to X$ that is holomorphic on a neighborhood of K (independent of p) and such that $f(p, \cdot)|_{X'}$ is holomorphic on X', and* (b) *$f(p, \cdot)$ is holomorphic on X for every $p \in P_0$, then there is a homotopy $f_t : P \times X \to Z$ ($t \in [0, 1]$), with $f_0 = f$, such that f_t enjoys properties* (a) *and* (b) *for all $t \in [0, 1]$, and*

(i) *$f_1(p, \cdot)$ is holomorphic on X for all $p \in P$,*
(ii) *f_t is uniformly close to f on $P \times K$ for all $t \in [0, 1]$, and*
(iii) *$f_t = f$ on $(P_0 \times X) \cup (P \times X')$ for all $t \in [0, 1]$.*

If in addition $f(p, \cdot)$ is holomorphic on a neighborhood of X' for all $p \in P$, then the homotopy f_t can be chosen fixed to a given finite order along X'.

In the special case when $Z \to X$ is a holomorphic fiber bundle whose fiber is a complex homogeneous manifold, Theorem 1.1 is due to Grauert [6]. In 1989 Mikhael Gromov published an influential paper [7] in which he obtained Theorem 1.1 when $Z \to X$ is a fiber bundle over a Stein manifold X whose fiber Y is *elliptic* in the sense that it admits a *dominating holomorphic spray* — a triple (E, π, s) consisting of a holomorphic vector bundle $\pi : E \to Y$ and a holomorphic map $s : E \to Y$ such that for each $y \in Y$ we have $s(0_y) = y$ and $(\mathrm{d}s)_{0_y}(E_y) = \mathrm{T}_y Y$. It is easily seen that ellipticity implies CAP, but the converse implication is not known. In the same paper Gromov asked whether a Runge type approximation property for holomorphic maps $\mathbb{C}^n \to Y$ might suffice to infer the Oka principle [7, 3.4.(D), p. 881]. Theorem 1.1 gives a positive answer to Gromov's question and shows that all Oka properties considered in the literature are equivalent to each other. Hence this is an opportune moment to introduce the following class of complex manifolds:

Definition 1.2. A complex manifold Y is said to be an *Oka manifold* if it enjoys CAP or, equivalently, the parametric Oka property for maps of all reduced Stein spaces to Y.

Theorem 1.1 indicates that the class of Oka manifolds is dual to the class of Stein manifold in a sense that can be made precise by means of abstract homotopy theory (see Lárusson [8,9]).

Examples of Oka manifolds can be found in the papers [7,2,1]. The following result is useful for finding new examples:

Corollary 1.3. *(Cf. [2, Theorem 1.4].) Assume that E and B are complex manifolds. If $\pi : E \to B$ is a holomorphic fiber bundle whose fiber Y is an Oka manifold, then E is an Oka manifold if and only if B is an Oka manifold.*

2. A characterization of the parametric Oka property

We recall from [2] the precise definition of CAP and its parametric analogue, PCAP. Let $z = (z_1, \ldots, z_n)$ be complex coordinates on $\mathbb{C}^n$, with $z_j = x_j + \mathrm{i}y_j$. A *special convex pair* (K, L) in $\mathbb{C}^n$ consists of a cube $L = \{z \in \mathbb{C}^n\colon\ |x_j| \leqslant a_j,\ |y_j| \leqslant b_j,\ j = 1, \ldots, n\}$ and a compact convex set $K = \{z \in L\colon\ y_n \leqslant h(z_1, \ldots, z_{n-1}, x_n)\}$, where h is a continuous concave function with values in $(-b_n, b_n)$.

We say that a map is holomorphic on a compact set if it is holomorphic in an open neighborhood of that set.

Definition 2.1. A complex manifold Y enjoys CAP if for each special convex pair (K, L) in $\mathbb{C}^n$, every holomorphic map $f : K \to Y$ can be approximated uniformly on K by holomorphic maps $L \to Y$.

Y enjoys the *Parametric Convex Approximation Property* (PCAP) for a pair of topological spaces $P_0 \subset P$ if for every special convex pair (K, L), a continuous map $f : P \times L \to Y$ such that $f(p, \cdot) : L \to Y$ is holomorphic for every $p \in P_0$, and is holomorphic on K for every $p \in P$, can be approximated uniformly on $P \times K$ by continuous maps $\tilde{f} : P \times L \to Y$ such that $\tilde{f}(p, \cdot)$ is holomorphic on L for all $p \in P$, and $\tilde{f} = f$ on $P_0 \times L$.

The following result was proved in [2,1,3]:

Theorem 2.2. (a) *If Y enjoys CAP, then it enjoys the basic Oka property* (*the conclusion of Theorem* 1.1 *for P a singleton and $P_0 = \emptyset$*).

(b) *If Y enjoys PCAP for a pair of compact Hausdorff spaces $P_0 \subset P$, then it also satisfies the conclusion of Theorem* 1.1 *for the same pair.*

Hence Theorem 1.1 follows from the following result:

Theorem 2.3. *If Y enjoys CAP, then it also enjoys PCAP for all pairs $P_0 \subset P$ of compact parameter spaces contained in a Euclidean space $\mathbb{R}^m$.*

3. Proof of Theorem 2.3

Assume that $P_0 \subset P$ are compact sets in $\mathbb{R}^m \subset \mathbb{C}^m$, (K, L) is a special convex pair in $\mathbb{C}^n$, $U \supset K$ and $V \supset L$ are open convex neighborhoods of K resp. L in $\mathbb{C}^n$, and $f : P \times V \to Y$ is such that $f(p, \cdot) : V \to Y$ is holomorphic for every $p \in P_0$, and it is holomorphic on U for every $p \in P$.

By using a Tietze extension theorem for Banach-valued maps as in [4, Proposition 4.4] and shrinking the set $V \supset L$ we can assume that $f(p, \cdot)$ is holomorphic on V for all p in a neighborhood $P_0' \subset \mathbb{C}^m$ of P_0.

We may assume that $0 \in \mathbb{C}^n$ belongs to $\operatorname{Int} K$. Choose a continuous function $\tau : P \to [0, 1]$ such that $\tau = 0$ on P_0 and $\tau = 1$ on $P \backslash P_0'$. Set

$$f_t(p, z) = f\big(p, \big(1 - (1 - t)\tau(p)\big)z\big) \in Y, \quad p \in P,\ z \in V,\ t \in [0, 1].$$

Then f_t has the same properties as $f = f_1$, the homotopy is fixed for $p \in P_0$, and the map $f_0(p, \cdot)$ is holomorphic on V for all $p \in P$.

We shall follow Gromov's proof of the h-Runge theorem in [7], using local sprays over Stein domains covering the graph of the homotopy f_t, along with the basic Oka property of Y (which is implied by CAP in view of Theorem 2.2(a)). Set $Z = \mathbb{C}^m \times \mathbb{C}^n \times Y$. For every $t \in [0, 1]$ let

$$F_t(p, z) = \big(p, z, f_t(p, z)\big), \quad \Sigma_t = F_t(P \times K) \subset Z.$$

We also set $S_0 = F_0(P \times L)$. By [5, Corollary 2.2] the sets Σ_t and S_0 are Stein compact in Z. Hence there are numbers $0 = t_0 < t_1 < \cdots < t_N = 1$ and Stein domains $\Omega_j \subset Z$ such that

$$\Sigma_t \subset \Omega_j \quad \text{when } t_j \leqslant t \leqslant t_{j+1} \quad \text{and} \quad j = 0, 1, \ldots, N - 1. \tag{1}$$

Let $\pi_Y : Z \to Y$ denote the projection onto Y, and let $E = \pi_Y^*(TY)$ denote the pull-back of the tangent bundle of Y to Z. By standard techniques we obtain for every Stein domain $\Omega \subset Z$ a Stein neighborhood $W \subset E|_\Omega$ of the zero section $\Omega \subset E|_\Omega$ and a holomorphic map $s : W \to Z$ that maps the fiber $W_{(\zeta,z,y)}$ over $(\zeta, z, y) \in Z$ biholomorphically onto a neighborhood of this point in $\{(\zeta, z)\} \times Y$ such that s preserves the zero section Ω of $E|_\Omega$. In short, s is a local fiber dominating spray in the sense of Gromov [7]. We may assume that W is Runge in $E|_\Omega$ and its fibers are convex (in the fibers of E). Since $E|_\Omega$ is Stein and Y enjoys CAP (and hence the basic Oka property by Theorem 2.2(a)), s can be approximated uniformly on compacts in W by a global spray $\tilde{s} : E|_\Omega \to Z$ that agrees with s to the second order along the zero section Ω.

This shows that after refining our subdivision $\{t_j\}$ of $[0, 1]$ and shrinking the set $U \supset K$ there are Stein domains $\Omega_0, \ldots, \Omega_{N-1}$ as in (1), sprays $s_j : E|_{\Omega_j} \to Z$, and homotopies of z-holomorphic sections ξ_t $(t \in [t_j, t_{j+1}])$ of the restricted bundle $E|_{F_{t_j}(P \times U)}$ such that ξ_{t_j} is the zero section, $\xi_t(p, \cdot)$ is independent of t when $p \in P_0$ (hence it is the zero section), and

$$s_j \circ \xi_t \circ F_{t_j} = F_t \quad \text{on } P \times U, \quad t \in [t_j, t_{j+1}].$$

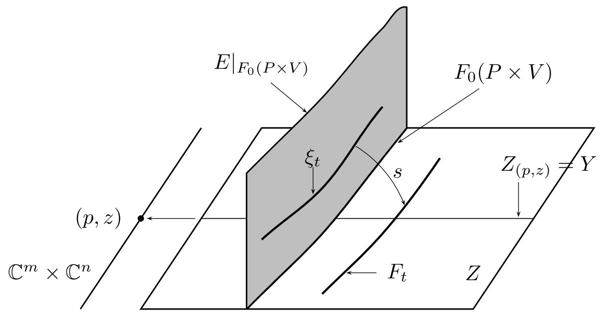

Fig. 1. Lifting sections F_t to the spray bundle $E|_{F_0(P\times V)}$.

Furthermore, the existence of such liftings ξ_t is stable under sufficiently small perturbations of the homotopy F_t (see Fig. 1).

Consider the homotopy of sections ξ_t of $E|_{F_0(P\times U)}$ for $t \in [0, t_1]$. By the Oka–Weil theorem we can approximate ξ_t uniformly on $P \times K$ by z-holomorphic sections $\tilde{\xi}_t$ of $E|_{F_0(P\times V')}$ for an open convex set $V' \subset \mathbb{C}^n$ with $L \subset V' \subset V$. (This parametric version of the Oka–Weil theorem is obtained by using a continuous partition of unity with respect to the parameter.) Further, we may choose $\tilde{\xi}_t = \xi_t$ for $t = 0$ and on $P_0 \times V'$.

By [5, Corollary 2.2] there is a Stein neighborhood $\Omega \subset Z$ of S_0 such that Σ_0 is $\mathcal{O}(\Omega)$-convex. Hence $E|_{\Sigma_0}$ is exhausted by $\mathcal{O}(E|_\Omega)$-convex compact sets. Since $E|_\Omega$ is Stein and Y enjoys CAP, the spray s can be approximated on the range of the homotopy $\{\xi_t\colon\ t \in [0, t_1]\}$ by a spray $\tilde{s}: E|_\Omega \to Z$ that agrees with s to the second order along the zero section. The maps

$$\tilde{f}_t = \pi_Y \circ \tilde{s} \circ \tilde{\xi}_t \circ F_0 : P \times V' \to Y, \quad t \in [0, t_1]$$

are then z-holomorphic on $V' \supset L$, and they approximate f_t uniformly on $P \times K$. If the approximation is sufficiently close, we obtain a new homotopy $\{f_t\colon\ t \in [0, 1]\}$ that agrees with $\tilde{f}_t$ for $t \in [0, t_1]$ (hence is z-holomorphic on L for these values of t), and that agrees with the initial homotopy for $t \in [t_1', 1]$ for some $t_1' > t_1$ close to t_1.

We now repeat the same argument with the parameter interval $[t_1, t_2]$, using f_{t_1} as the new reference map (that is z-holomorphic over L). This gives us a new homotopy that is z-holomorphic on L for $t \in [0, t_2]$.

After finitely many steps of this kind we obtain a desired homotopy whose final map $\tilde{f}$ at $t = 1$ satisfies the conclusion of Theorem 2.3.

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