Partial Differential Equations

Uniqueness of unbounded solutions of the Lagrangian mean curvature flow equation for graphs

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Abstract

We observe that the comparison result of Barles–Biton–Ley for viscosity solutions of a class of nonlinear parabolic equations can be applied to a geometric fully nonlinear parabolic equation which arises from the graphic solutions for the Lagrangian mean curvature flow. To cite this article: J. Chen, C. Pang, C. R. Acad. Sci. Paris, Ser. I 347 (2009).

1. Introduction

We consider the question of uniqueness for the following fully nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{n} \arctan \lambda_j$$

with initial condition $u(x, 0) = u_0(x)$, where $u$ is a function from $\mathbb{R}^n$ to $\mathbb{R}$ and $\lambda_j$'s are the eigenvalues of the Hessian $D^2u$. This equation arises naturally from geometry. In fact, when $u$ is a regular solution to (1), it is known that the graph $(x, Du(x, t))$ evolves by the mean curvature flow and it is a Lagrangian submanifold in $\mathbb{R}^n \times \mathbb{R}^n$ with the standard symplectic structure, for each $t$ (cf. [5,6]). For a smooth stationary solution to (1), the graph of its gradient is a Lagrangian submanifold with zero mean curvature in $\mathbb{R}^{2n}$. Recently, a smooth longtime entire solution to (1) has been constructed in [2] assuming a certain bound on the Lipschitz norm of $Du_0$. 

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Barles, Biton and Ley have obtained a very useful general comparison result (Theorem 2.1 in [1]) for the viscosity solutions of a class of fully nonlinear parabolic equations, as well as existence result (Theorem 3.1 in [1]). In particular, they showed that (1) admits a unique longtime continuous viscosity solution for any continuous function $u_0$ in $\mathbb{R}$ when $n = 1$.

In this short note, we observe, via elementary methods, that the hypotheses in the general theorems in [1] are valid for the geometric evolution equation (1) in general dimensions. The result is the following:

**Theorem 1.1.** Let $u$ and $v$ be an upper semicontinuous and a lower semicontinuous viscosity subsolution and supersolution to (1) in $\mathbb{R}^n \times [0, T)$ respectively. If $u(x, 0) \leq v(x, 0)$ for all $x \in \mathbb{R}^n$, then $u \leq v$ in $\mathbb{R}^n \times [0, T)$. In particular, for any continuous function $u_0$ in $\mathbb{R}^n$, there is a unique continuous viscosity solution to (1) in $\mathbb{R}^n \times [0, \infty)$.

2. Hypotheses (H1) and (H2)

We now describe the assumptions in the comparison and existence results in [1].

Let $S_n$ be the linear space of real $n \times n$ symmetric matrices. If $X \in S_n$, there exists an orthogonal matrix $P$ such that $X = P\Lambda P^T$ where $\Lambda$ is the diagonal matrix with diagonal entries consist of eigenvalues of $X$. Let $\Lambda^+$ be the diagonal matrix obtained by replacing the negative eigenvalues in $\Lambda$ with 0’s. Define $X^+ = P\Lambda^+ P^T$.

Consider a continuous function $F$ from $\mathbb{R}^n \times [0, T] \times \mathbb{R}^n \times S_n$ to $\mathbb{R}$. The following assumptions on $F$ are necessary to apply the results in [1]:

(H1) For any $R > 0$, there exists a function $m_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that $m_R(0^+) = 0$ and

$$F(y, t, \eta(x-y), Y) - F(x, t, \eta(x-y), X) \leq m_R(\eta|x - y|^2 + |x - y|)$$

for all $x, y \in \overline{B}(0, R)$ and $t \in [0, T]$, whenever $X, Y \in S_n$ and $\eta > 0$ satisfy

$$-3\eta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leq \begin{pmatrix} X & 0 & -Y \\ 0 & -I & Y \\ -Y & Y & I \end{pmatrix} \leq 3\eta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

(H2) There exist $0 < \alpha < 1$ and constants $K_1 > 0$ and $K_2 > 0$ such that

$$F(x, t, p, X) - F(x, t, q, Y) \leq K_1 |p - q|((1 + |x|) + K_2 (\text{tr}(Y - X)^+)^\alpha)$$

for every $(x, t, p, q, X, Y) \in \mathbb{R}^n \times [0, T] \times \mathbb{R}^n \times S_n \times S_n$.

The operator $F$ is degenerate elliptic if (H2) holds.

**Theorem 2.1 (Barles–Biton–Ley).** Let $u$ and $v$ be an upper semicontinuous viscosity subsolution and a lower semicontinuous viscosity supersolution respectively of

$$\frac{\partial u}{\partial t} + F(x, t, Du, D^2 u) = 0 \quad \text{in } \mathbb{R}^n \times [0, T),$$

$$u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^n .$$

Assume that (H1) and (H2) hold for $F$. Then

1. If $u(\cdot, 0) \leq v(\cdot, 0)$ in $\mathbb{R}^n$, then $u \leq v$ in $\mathbb{R}^n \times [0, T)$.
2. If $u_0 \in C(\mathbb{R}^n)$ there is a unique continuous viscosity solution in $\mathbb{R}^n \times [0, \infty)$.

We now present the proof of Theorem 1.1.

**Proof.** We define $F : S_n \to \mathbb{R}$ by

$$F(X) = -i \log \frac{\det(I + iX)}{\det(I + X^2)} = -\frac{i}{2} \log \frac{\det(I + iX)}{\det(I - iX)} .$$

(2)
That $F$ takes real values follows easily from

$$F(X) = \frac{i}{2} \log \frac{\det(I - iX)}{\det(I + iX)} = F(X).$$

Note that $F(D^2u)$, by diagonalizing $D^2u$ at a point, is equal to $\sum \arctan \lambda_j$. Therefore the flow (1) can be written as $u_t + (-F(D^2u)) = 0$.

Since $F(x, t, p, X) = F(X)$ is independent of $x$, the right-hand side of the inequality for $F$ in (H1) must be zero, namely $m_R = 0$. By multiplying an arbitrary vector $(\xi, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and its transpose to the second matrix inequality in (H1), we see that $X \preceq Y$. Therefore, in order to establish (H1) it suffices to show:

(H1′) For any $X, Y \in S_n$, if $X \succeq Y$ then $F(X) \succeq F(Y)$.

For any $X, Y \in S_n$ and $t \in [0, 1]$, define

$$f_{XY}(t) = F(tX + (1-t)Y).$$

We will show that $f_{XY}(t)$ is nondecreasing in $t \in [0, 1]$ and then (H1′) will follow as $f_{XY}(0) = F(Y)$ and $f_{XY}(1) = F(X)$. Set

$$A = I + i(tX + (1-t)Y)$$

and

$$B = I - i(tX + (1-t)Y).$$

Then

$$f_{XY}(t) = -\frac{i}{2} (\log \det A - \log \det B).$$

It follows that $AB = BA$ and

$$(A^{-1} + B^{-1}) \cdot \frac{AB}{2} = \frac{A + B}{2} = I.$$

Note that both $A$ and $B$ are invertible matrices for all $t \in [0, 1]$. Hence, by using the formula $\partial_t \ln \det G = \text{tr}(G^{-1} \partial_t G)$ for $G(t) \in GL(n, \mathbb{R})$, we have

$$f'_{XY}(t) = -\frac{i}{2} \text{tr}(A^{-1} \cdot \partial_t A - B^{-1} \cdot \partial_t B)$$

$$= -\frac{i}{2} \text{tr}((A^{-1} + B^{-1}) \cdot i(X - Y))$$

$$= \text{tr}((I + (tX + (1-t)Y)^2)\cdot (X - Y)).$$

(3)

Since $tX + (1-t)Y$ is real symmetric, the matrix

$$C = I + (tX + (1-t)Y)^2$$

is positive definite, hence so is $C^{-1}$. There exists a matrix $Q \in GL(n, \mathbb{R})$ such that $C = QQ^T$. By the assumption $X \succeq Y$, we have

$$\text{tr}(C^{-1}(X - Y)) = \text{tr}(Q \cdot Q^T (X - Y))$$

$$= \text{tr}(Q^T (X - Y) \cdot Q)$$

$$\geq 0$$

since $Q^T (X - Y)Q$ is positive semidefinite. Therefore, we have shown that (H1) is valid for $F$ defined in (2).

As $F(x, t, p, X)$ is independent of $p$, (H2) reads: there exist constants $K > 0$ and $0 < \alpha < 1$ such that $F(X) - F(Y) \leq K (\text{tr}(X - Y)^+)^\alpha$ for all $X, Y \in S_n$. For any $X, Y \in S_n$, integrating (3) leads to

$$F(X) - F(Y) = \int_0^1 \text{tr}(C^{-1}(X - Y)) \, dt.$$  (4)
For $X - Y \in S_n$ there exists an orthogonal matrix $P$ such that $X - Y = P \Lambda P^T$ where the diagonal matrix $\Lambda$ has diagonal entries $\lambda_1, \ldots, \lambda_n$. Let $\lambda_j^+ = \max\{\lambda_j, 0\}$. Since $0 < C^{-1} \leq I$, we have $0 < P^T C^{-1} P \leq I$. If $c_{jj}$ denote the diagonal entries of $P^T C^{-1} P$ for $j = 1, \ldots, n$, then $c_{jj} = \langle P^T C^{-1} P e_j, e_j \rangle$ where $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. It follows that $0 < c_{jj} \leq 1$ for $j = 1, \ldots, n$. Then

$$
\text{tr}(C^{-1}(X - Y)) = \text{tr}(P^T C^{-1} P \cdot P^T (X - Y) P) \\
= \text{tr}(P^T C^{-1} P \cdot \Lambda) \\
= \sum c_{jj} \lambda_j \\
\leq \sum \lambda_j^+ \\
= \text{tr}(X - Y)^+.
$$

Substituting the above inequality into (4) implies: for any $X, Y \in S_n$ we have $F(X) - F(Y) \leq \text{tr}(X - Y)^+$. Because $\arctan x$ is in $(-\pi/2, \pi/2)$, we have $F(X) - F(Y) < n\pi$. For any constant $\alpha$ with $0 < \alpha < 1$, if $\text{tr}(X - Y)^+ \leq 1$ then

$$
F(X) - F(Y) \leq \text{tr}(X - Y)^+ \leq n\pi [\text{tr}(X - Y)^+]^\alpha
$$

and if $\text{tr}(X - Y)^+ > 1$ then

$$
F(X) - F(Y) \leq n\pi \leq n\pi [\text{tr}(X - Y)^+]^\alpha.
$$

Therefore, (H2) holds for $K_2 = n\pi$ and any constants $K_1 > 0$ and $\alpha$ with $0 < \alpha < 1$.

Now Theorem 1.1 follows immediately from Theorem 2.1. \qed

We notice that (H1'), for the operator $F(X) = \sum \arctan \lambda_j(X)$, also follows from the basic fact (cf. p. 182 in [4]): Suppose that $X, Y \in S_n$ and the eigenvalues $\lambda_j$’s of $X$ and $\mu_j$’s of $Y$ are in descending order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. If $X \geq Y$, then $\lambda_j \geq \mu_j$ for $j = 1, \ldots, n$.

We also mention the uniqueness of viscosity solutions of the Cauchy–Dirichlet problem for (1). Note that the operator $F(X) = \sum \arctan \lambda_j(X)$ satisfies (H1’) which is exactly the fundamental monotonicity condition (0.1) for $-F$ in [3], therefore $-F$ is proper in the sense of [3] (cf. p. 2 in [3]). As (H1) holds, Theorem 8.2 in [3] is valid for (1):

**Theorem 2.2.** The continuous viscosity solution to the following Cauchy–Dirichlet problem is unique:

$$
u_t = \sum_{j=1}^n \arctan \lambda_j, \quad \text{in } (0, T) \times \Omega,$$

$$u(t, x) = 0, \quad \text{for } 0 \leq t < T \text{ and } x \in \partial \Omega,$$

$$u(0, x) = \psi(x), \quad \text{for } x \in \bar{\Omega},$$

where $\lambda_j$’s are the eigenvalues of $D^2 u$, $\Omega \subset \mathbb{R}^n$ is open and bounded and $T > 0$ and $\psi \in C(\bar{\Omega})$. If $u$ is an upper semicontinuous viscosity subsolution and $v$ is a lower semicontinuous viscosity supersolution of the Cauchy–Dirichlet problem, then $u \leq v$ on $[0, T) \times \Omega$.

Note that the initial-boundary conditions for the subsolution and supersolution are: $u(x, t) \leq 0 \leq v(x, t)$ for $t \in [0, T)$ and $x \in \partial \Omega$ and $u(x, 0) \leq \psi(x) \leq v(x, 0)$ for $x \in \bar{\Omega}$.

**References**