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Partial Differential Equations

Time decay for hyperbolic equations with homogeneous symbols

Tokio Matsuyama [a], Michael Ruzhansky [b]

[a] Department of Mathematics, Tokai University, Hiratsuka, Kanagawa 259-1292, Japan
[b] Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, United Kingdom

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Abstract

The aim of this Note is to present dispersive estimates for strictly hyperbolic equations with time dependent coefficients that have integrable derivatives. We will relate the time decay rate of L^p–L^q norms of solutions to certain geometric indices associated to the characteristics of the limiting equation. Results will be applied to the global solvability of Kirchhoff type equations with small data, and to the dispersive estimates for their solutions. ***To cite this article: T. Matsuyama, M. Ruzhansky, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Estimations en temps pour des équations hyperboliques à symboles homogènes. Dans cette Note, nous établissons des estimations dispersives des solutions d'équations strictement hyperboliques à coefficients dépendant du temps et à dérivées intégrables. Nous estimons le taux de décroissance en temps des normes des solutions en fonction d'indices géométriques associés aux caractéristiques de l'équation limite. Les résultats sont appliqués à la résolvabilité globale des équations de type Kirchhoff à données petites et à des estimations dispersives des solutions. ***Pour citer cet article : T. Matsuyama, M. Ruzhansky, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Version française abrégée

Dans cette Note nous considérons le problème de Cauchy pour une équation strictement hyperbolique d'ordre de m à coefficients dépendant du temps :

$$L(t, D_t, D_x)u \equiv D_t^m u + \sum_{j\leqslant m-1, |\nu|+j=m} a_{\nu,j}(t) D_x^\nu D_t^j u = 0, \quad t \neq 0, \tag{1}$$

à données de Cauchy,

$$D_t^k u(0,x) = f_k(x) \in C_0^\infty\big(\mathbb{R}^n\big), \quad k = 0, 1, \ldots, m-1, \quad x \in \mathbb{R}^n, \tag{2}$$

E-mail addresses: tokio@keyaki.cc.u-tokai.ac.jp (T. Matsuyama), m.ruzhansky@imperial.ac.uk (M. Ruzhansky).

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où $D_t = \frac{1}{i}\frac{\partial}{\partial t}$ et $D_x^\nu = (\frac{1}{i}\frac{\partial}{\partial x_1})^{\nu_1}\cdots(\frac{1}{i}\frac{\partial}{\partial x_n})^{\nu_n}$, $i=\sqrt{-1}$, pour $\nu=(\nu_1,\dots,\nu_n)$. Nous supposons que toutes les dérivées temporelles des coefficients $a_{\nu,j}(t)$ jusqu'à l'ordre $m-1$ sont bornés et continus sur $\mathbb{R}$, et que les $a_{\nu,j}(t)$ satisfont à :

$$\partial_t^k a_{\nu,j}(t)\in L^1(\mathbb{R}) \quad \text{pour tout } \nu,\ j \text{ avec } |\nu|+j=m,\ \text{et } k=1,\dots,m-1. \tag{3}$$

D'après la définition standard des équations réguliérement hyperboliques, par exemple Mizohata [8], nous supposons aussi que le symbole de l'opérateur différentiel $L(t,D_t,D_x)$ a ses racines réelles et distinctes $\varphi_1(t;\xi),\dots,\varphi_m(t;\xi)$ pour $\xi\neq 0$, et que

$$L(t,\tau,\xi)=\big(\tau-\varphi_1(t;\xi)\big)\cdots\big(\tau-\varphi_m(t;\xi)\big), \quad \text{avec} \inf_{j\neq k,|\xi|=1,t\in\mathbb{R}}\big|\varphi_j(t;\xi)-\varphi_k(t;\xi)\big|>0. \tag{4}$$

Le problème de l'évaluation des estimations de décroissance en temps pour la norme L^p–L^q du propagateur pour (1)–(2) revient à estimer les asymptotiques des intégrales oscillants apparaissant dans la représentation de la solution $u(t,x)$. Dans ce but, nous devons d'abord obtenir de telles représentations ce qui peut etre fait en développant la méthode de l'intégration asymptotique. Ensuite nous identifirons les quantités qui interviennent dans le taux de décroissance en temps de la solution.

Tout d'abord nous notons qu'on peut montrer que sous les conditions (3) et (4), pour tout $l=1,\dots,m$, il existe des limites $\varphi_l^\pm(\xi)=\lim_{t\to\pm\infty}\varphi_l(t,\xi)$, qui sont des fonctions positivement homogènes d'ordre un. Notons $\Sigma_{\varphi_l^\pm}=\{\xi\in\mathbb{R}^n\colon \varphi_l^\pm(\xi)=1\}$. Si $\Sigma=\Sigma_{\varphi_l^\pm}$ est convexe, nous pouvons définir son indice convexe de Sugimoto $\gamma(\Sigma)$ de la manière suivante. Nous posons :

$$\gamma(\Sigma):=\sup_{\sigma\in\Sigma}\sup_P \gamma(\Sigma;\sigma,P),$$

où P est un plan contenant le vecteur normal à Σ à σ, et $\gamma(\Sigma;\sigma,P)$ est l'ordre du contact entre la ligne $T_\sigma\cap P$ (où T_σ est le plan tangeant à σ), et la courbe $\Sigma\cap P$. Dans le cas où la surface $\Sigma=\Sigma_{\varphi_l^\pm}$ n'est pas convexe, nous définissons son indice de Sugimoto non convexe $\gamma_0(\Sigma)$ par

$$\gamma_0(\Sigma):=\sup_{\sigma\in\Sigma}\inf_P \gamma(\Sigma;\sigma,P),$$

où P et σ sont analogues à celles du cas convexe.

Théorème 0.1. *Supposons* (3)–(4). *Alors la solution* $u(t,x)$ *de* (1) *vérifie les estimations suivantes* :

(i) *Supposons que* $\Sigma_{\varphi_\ell^\pm}=\{\xi\in\mathbb{R}^n\colon \varphi_\ell^\pm(\xi)=1\}$ *est convexe pour tout* $\ell=1,\dots,m$, *et désignons* $\gamma:=\max_{\ell=1,\dots,m}\gamma(\Sigma_{\varphi_\ell^\pm})$. *De plus, supposons que* $(1+|t|)^r\partial_t^j a_{\nu,k}\in L^1(\mathbb{R})$ *pour* $1\leqslant r\leqslant[(n-1)/\gamma]+1$, $j=1,\dots,m-1$ *et pour tout* ν,k *avec* $|\nu|+k=m$; $1<p\leqslant 2\leqslant q<+\infty$ *et* $\frac{1}{p}+\frac{1}{q}=1$. *Alors, pour tout* $t\in\mathbb{R}$ *nous avons l'estimation,*

$$\big\|D_t^l D_x^\alpha u(t,\cdot)\big\|_{L^q(\mathbb{R}^n)}\leqslant C\big(1+|t|\big)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}\sum_{k=0}^{m-1}\big(\|f_k\|_{\dot L^p_{N_p+l+|\alpha|-k}(\mathbb{R}^n)}+\|f_k\|_{\dot L^p_{l+|\alpha|-k}(\mathbb{R}^n)}\big),$$

où $N_p=(n-\frac{n-1}{\gamma}+[\frac{n-1}{\gamma}]+1)(\frac{1}{p}-\frac{1}{q})$, $l=0,\dots,m-1$, *et pour tout multiindex* α.

(ii) *Supposons que* $\Sigma_{\varphi_\ell^\pm}$ *est non convexe pour certains* $\ell=1,\dots,m$, *et soit* $\gamma_0:=\max_{\ell=1,\dots,m}\gamma_0(\Sigma_{\varphi_\ell^\pm})$. *De plus, supposons que* $(1+|t|)\partial_t a_{\nu,k}\in L^1(\mathbb{R})$ *pour tout* ν,k *with* $|\nu|+k=m$. *Soit* $1<p\leqslant 2\leqslant q<+\infty$ *et* $\frac{1}{p}+\frac{1}{q}=1$. *Alors, pour tout* $t\in\mathbb{R}$ *nous avons l'estimation* :

$$\big\|D_t^l D_x^\alpha u(t,\cdot)\big\|_{L^q(\mathbb{R}^n)}\leqslant C\big(1+|t|\big)^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})}\sum_{k=0}^{m-1}\big(\|f_k\|_{\dot L^p_{N_p+l+|\alpha|-k}(\mathbb{R}^n)}+\|f_k\|_{\dot L^p_{l+|\alpha|-k}(\mathbb{R}^n)}\big),$$

où $N_p=(n-\frac{1}{\gamma_0}+1)(\frac{1}{p}-\frac{1}{q})$, $l=0,\dots,m-1$, *et pour tout multiindex* α.

Théorème 0.1 sera appliqués à la solvabilité globale des équations de type Kirchhoff à petites données et aux estimations dispersives pour leurs solutions.

1. Linear equations

In this Note we consider the Cauchy problem for the strictly hyperbolic equation of order m with coefficients dependent on time:

$$L(t, D_t, D_x)u \equiv D_t^m u + \sum_{j \leqslant m-1, |\nu|+j=m} a_{\nu,j}(t) D_x^\nu D_t^j u = 0, \quad t \neq 0, \tag{5}$$

with Cauchy data,

$$D_t^k u(0,x) = f_k(x) \in C_0^\infty(\mathbb{R}^n), \quad k = 0, 1, \ldots, m-1, \quad x \in \mathbb{R}^n, \tag{6}$$

where $D_t = \frac{1}{i}\frac{\partial}{\partial t}$ and $D_x^\nu = (\frac{1}{i}\frac{\partial}{\partial x_1})^{\nu_1} \cdots (\frac{1}{i}\frac{\partial}{\partial x_n})^{\nu_n}$, $i = \sqrt{-1}$, for $\nu = (\nu_1, \ldots, \nu_n)$. We will assume that all time-derivatives of the coefficients $a_{\nu,j}(t)$ up to the order $m-1$ are bounded and continuous on $\mathbb{R}$, and that each $a_{\nu,j}(t)$ satisfies:

$$\partial_t^k a_{\nu,j}(t) \in L^1(\mathbb{R}) \quad \text{for all } \nu, j \text{ with } |\nu| + j = m, \text{ and } k = 1, \ldots, m-1. \tag{7}$$

Following the standard definition of equations of the regularly hyperbolic type (e.g. Mizohata [8]), we will also assume that the symbol of the differential operator $L(t, D_t, D_x)$ has real and distinct roots $\varphi_1(t;\xi), \ldots, \varphi_m(t;\xi)$ for $\xi \neq 0$, and that

$$L(t,\tau,\xi) = \big(\tau - \varphi_1(t;\xi)\big) \cdots \big(\tau - \varphi_m(t;\xi)\big), \quad \text{with} \inf_{j \neq k, |\xi|=1, t \in \mathbb{R}} \big|\varphi_j(t;\xi) - \varphi_k(t;\xi)\big| > 0. \tag{8}$$

First we note, that it can be shown that under conditions (7) and (8), for all $l = 1, \ldots, m$, there exist limits $\varphi_l^\pm(\xi) = \lim_{t \to \pm\infty} \varphi_l(t,\xi)$, which are positively homogeneous of order one functions. We note that in the case of oscillation of coefficients the situation is more delicate, and essentially only results for certain second order equations are available (e.g. [9,10]). Let us denote $\Sigma_{\varphi_l^\pm} = \{\xi \in \mathbb{R}^n\colon \varphi_l^\pm(\xi) = 1\}$. If the set $\Sigma = \Sigma_{\varphi_l^\pm}$ is convex, we can define its convex Sugimoto index $\gamma(\Sigma)$ in the following way. We set:

$$\gamma(\Sigma) := \sup_{\sigma \in \Sigma} \sup_P \gamma(\Sigma; \sigma, P),$$

where P is a plane containing the normal to Σ at σ and $\gamma(\Sigma;\sigma,P)$ denotes the order of the contact between the line $T_\sigma \cap P$ (where T_σ is the tangent plane at σ), and the curve $\Sigma \cap P$. In the case when the level set $\Sigma = \Sigma_{\varphi_l^\pm}$ is not convex, we define its non-convex Sugimoto index $\gamma_0(\Sigma)$ by:

$$\gamma_0(\Sigma) := \sup_{\sigma \in \Sigma} \inf_P \gamma(\Sigma; \sigma, P),$$

where P and σ are the same as in the convex case. The indices γ and γ_0 have been introduced in the time-independent context in [12,13], and they are related to oscillation indices in the singularity theory (in e.g. [1]). We use the notation L_s^p for the standard Sobolev space with s derivatives in L^p, i.e. the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $(1-\Delta)^{s/2} f \in L^p$, and $\dot{L}_s^p$ for its homogeneous version, i.e. the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $(-\Delta)^{s/2} f \in L^p$. Detailed proofs of these theorems will appear in [7].

Theorem 1.1. *Assume* (7)–(8). *Then the solutions $u(t,x)$ of* (5) *satisfies the following estimates*:

(i) *Suppose that $\Sigma_{\varphi_\ell^\pm} = \{\xi \in \mathbb{R}^n\colon \varphi_\ell^\pm(\xi) = 1\}$ is convex for all $\ell = 1, \ldots, m$, and set $\gamma := \max_{\ell=1,\ldots,m} \gamma(\Sigma_{\varphi_\ell^\pm})$. In addition, suppose that $(1+|t|)^r \partial_t^j a_{\nu,k} \in L^1(\mathbb{R})$ for $1 \leqslant r \leqslant [(n-1)/\gamma] + 1$, $j = 1, \ldots, m-1$, and for all ν, k with $|\nu| + k = m$. Let $1 < p \leqslant 2 \leqslant q < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $t \in \mathbb{R}$ we have the estimate*:

$$\big\| D_t^l D_x^\alpha u(t,\cdot)\big\|_{L^q(\mathbb{R}^n)} \leqslant C\big(1+|t|\big)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})} \sum_{k=0}^{m-1} \big(\|f_k\|_{L^p_{N_p+l+|\alpha|-k}(\mathbb{R}^n)} + \|f_k\|_{\dot{L}^p_{l+|\alpha|-k}(\mathbb{R}^n)}\big),$$

where $N_p = (n - \frac{n-1}{\gamma} + [\frac{n-1}{\gamma}] + 1)(\frac{1}{p} - \frac{1}{q})$, $l = 0, \ldots, m-1$, and α is any multi-index.

(ii) *Suppose that* $\Sigma_{\varphi_\ell^\pm}$ *is non-convex for some* $\ell = 1, \dots, m$, *and set* $\gamma_0 := \max_{\ell=1,\dots,m} \gamma_0(\Sigma_{\varphi_\ell^\pm})$. *In addition, suppose that* $(1+|t|)\partial_t a_{\nu,k} \in L^1(\mathbb{R})$ *for all* ν, k *with* $|\nu| + k = m$. *Let* $1 < p \leqslant 2 \leqslant q < +\infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then for all* $t \in \mathbb{R}$ *we have the estimate*:

$$\left\| D_t^l D_x^\alpha u(t,\cdot) \right\|_{L^q(\mathbb{R}^n)} \leqslant C\big(1+|t|\big)^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})} \sum_{k=0}^{m-1} \big(\|f_k\|_{\dot{L}^p_{N_p+l+|\alpha|-k}(\mathbb{R}^n)} + \|f_k\|_{\dot{L}^p_{l+|\alpha|-k}(\mathbb{R}^n)} \big),$$

where $N_p = (n - \frac{1}{\gamma_0} + 1)(\frac{1}{p} - \frac{1}{q})$, $l = 0, \dots, m-1$, *and* α *is any multi-index.*

2. Nonlinear equations of Kirchhoff type

As a consequence, we can establish the well-posedness and asymptotic properties of the corresponding nonlinear equations of Kirchhoff type:

$$\widetilde{L}\big(t, D_t, D_x, \|\nabla u\|^2_{L^2(\mathbb{R}^n)}\big)u = D_t^m u + \sum_{\substack{|\nu|+j=m \\ j\leqslant m-1}} b_{\nu,j}\big(\|\nabla u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\big) D_x^\nu D_t^j u = 0, \tag{9}$$

for $t \neq 0$, with the initial condition,

$$D_t^k u(0,x) = f_k(x), \quad k = 0, 1, \dots, m-1, \quad x \in \mathbb{R}^n. \tag{10}$$

We will assume that the symbol of the differential operator $\widetilde{L}(t, D_t, D_x, \|\nabla u\|^2_{L^2(\mathbb{R}^n)})$ has real and distinct roots $\tilde{\varphi}_1(t,s;\xi), \dots, \tilde{\varphi}_m(t,s;\xi)$ for $\xi \neq 0$ and $0 \leqslant s \leqslant \delta$ with $\delta > 0$, i.e.,

$$\widetilde{L}(t,\tau,\xi,s) = \big(\tau - \tilde{\varphi}_1(t,s;\xi)\big) \cdots \big(\tau - \tilde{\varphi}_m(t,s;\xi)\big), \tag{11}$$

$$\inf_{\substack{|\xi|=1, t\in\mathbb{R}, s\in[0,\delta] \\ j\neq k}} \big|\tilde{\varphi}_j(t,s;\xi) - \tilde{\varphi}_k(t,s;\xi)\big| > 0. \tag{12}$$

The following results extend some of those in [2–6] and [15] to the setting of higher order equations, at the same time improving the indices in regularity assumptions. Full analysis will appear in [7], where we also develop the asymptotic integration methods in the PDE context (for the ODE setting see e.g. [14]). This yields a representation for solutions in the form of oscillatory integrals for which we carry out L^p–$L^{p'}$ estimates (for a survey on L^p–L^p estimates see e.g. [11]). We denote $L^2_s(\mathbb{R}^n)$ by $H^s(\mathbb{R}^n)$. We have the following global existence result with small data, which ensures the existence of global solutions for Theorem 2.2.

Theorem 2.1. *Let* $n \geqslant 1$ *and* $m \geqslant 2$. *Suppose* (11)–(12), *and suppose that all* $b_{\nu,j}(s)$ *and all of their derivatives up to the order* $m-1$ *are bounded on* $[0,\delta]$, *for* $\delta > 0$ *as in* (12). *Assume that* $f_k \in H^{m-k-1/2}(\mathbb{R}^n)$, $k = 0, \dots, m-1$, *satisfy*:

$$\sum_{k=0}^{m-1} \left\| \langle x \rangle^\varkappa f_k \right\|_{H^{m-k-1/2}(\mathbb{R}^n)} \ll 1 \quad \text{for some } \varkappa \geqslant 2. \tag{13}$$

Then (9)–(10) *has a unique solution* $u = u(t,x) \in \bigcap_{k=0}^m C^k(\mathbb{R}; H^{m-k-1/2}(\mathbb{R}^n))$.

Consequently, using the representation of solution for the limiting linearised problem and estimates of Theorem 1.1, we also obtain the dispersive estimates for solutions to the nonlinear problem (9)–(10).

Theorem 2.2. *Let* $n \geqslant 1$ *and* $m \geqslant 2$, *and let* $u(t,x)$ *be the solution to* (9)–(10) *from Theorem* 2.1 *with data satisfying* (13). *Then we have the following assertions*:

(i) *Suppose that* $\Sigma_{\tilde{\varphi}_\ell^\pm} = \{\xi \in \mathbb{R}^n\colon \tilde{\varphi}_\ell^\pm(\xi) = 1\}$ *is convex for all* $\ell = 1, \dots, m$, *and set* $\gamma := \max_{\ell=1,\dots,m} \gamma(\Sigma_{\tilde{\varphi}_\ell^\pm})$. *Let* $\varkappa > [(n-1)/\gamma] + 2$, *and* $1 < p \leqslant 2 \leqslant q < +\infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then for all* $t \in \mathbb{R}$ *we have the estimate*:

$$\left\| D_t^l D_x^\alpha u(t,\cdot) \right\|_{L^q(\mathbb{R}^n)} \leqslant C\big(1+|t|\big)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})} \sum_{k=0}^{m-1} \big(\|f_k\|_{\dot{L}^p_{N_p+l+|\alpha|-k}(\mathbb{R}^n)} + \|f_k\|_{\dot{L}^p_{l+|\alpha|-k}(\mathbb{R}^n)} \big),$$

where $N_p = (n - \frac{n-1}{\gamma} + [\frac{n-1}{\gamma}] + 1)(\frac{1}{p} - \frac{1}{q})$, $l = 0, \dots, m-1$, *and* α *is any multi-index.*

(ii) *Suppose that* $\Sigma_{\tilde{\varphi}_\ell^\pm}$ *is non-convex for some* $\ell = 1, \dots, m$, *and set* $\gamma_0 := \max_{\ell=1,\dots,m} \gamma_0(\Sigma_{\tilde{\varphi}_\ell^\pm})$. *Let* $\varkappa > 2$, *and* $1 < p \leqslant 2 \leqslant q < +\infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then for all* $t \in \mathbb{R}$ *we have the estimate*:

$$\left\| D_t^l D_x^\alpha u(t, \cdot) \right\|_{L^q(\mathbb{R}^n)} \leqslant C \big(1 + |t|\big)^{-\frac{1}{\gamma_0}(\frac{1}{p} - \frac{1}{q})} \sum_{k=0}^{m-1} \big(\|f_k\|_{\dot{L}^p_{N_p+l+|\alpha|-k}(\mathbb{R}^n)} + \|f_k\|_{\dot{L}^p_{l+|\alpha|-k}(\mathbb{R}^n)} \big),$$

where $N_p = (n - \frac{1}{\gamma_0} + 1)(\frac{1}{p} - \frac{1}{q})$, $l = 0, \dots, m-1$, *and* α *is any multi-index.*

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