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Uniform null controllability of the heat equation with rapidly oscillating periodic density

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Abstract

We consider the heat equation with fast oscillating periodic density, and an interior control in a bounded domain. First, we prove sharp convergence estimates depending explicitly on the initial data for the corresponding uncontrolled equation; these estimates are new, and their proof relies on a judicious smoothing of the initial data. Then we use those estimates to prove that the original equation is uniformly null controllable, provided a carefully chosen extra vanishing interior control is added to that equation. This uniform controllability result is the first in the multidimensional setting for the heat equation with oscillating density. Finally, we prove that the sequence of null controls converges to the optimal null control of the limit equation when the period tends to zero. ***To cite this article: L. Tebou, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Contrôlabilité à zéro uniforme de l'équation de la chaleur avec densité périodique rapidement oscillante. Nous considérons l'équation de la chaleur avec densité périodique rapidement oscillante, et un contrôle interne dans un domaine borné. Nous établissons d'abord des estimations fines de convergence dependant explicitement de la donnée initiale pour l'équation non contrôlée ; ces estimations sont nouvelles, et leur démonstration repose sur une régularisation judicieuse de la donnée initiale. Puis nous utilisons ces estimations pour démontrer que l'équation initiale est uniformément contrôlable à zéro, pourvu q'un contrôle interne, supplémentaire, evanescent, et convenablement choisi, soit ajouté à cette équation. Ce résultat de contrôlabilité uniforme est le premier dans le cadre multidimensionnel pour l'équation de la chaleur avec densité rapidement oscillante. Enfin nous montrons que la suite des contrôles converge vers le contrôle à zero optimal de l'équation limite lorsque la période tend vers zéro. ***Pour citer cet article : L. Tebou, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

On considère le problème de la contrôlabilité à zéro uniforme de l'equation de la chaleur avec densité périodique rapidement oscillante (voir (6)–(8) ci-après). On pose $D_\varepsilon = u_\varepsilon - u$, où u_ε et u sont les solutions de (10) et (11) respectivement. On établit les résultats suivants :

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Théorème 1. i) *Il existe une constante positive C telle que pour tous $u^0 \in H_0^1(\Omega)$, et $\varepsilon \in (0,1)$, on ait* :

$$\|D_\varepsilon\|_{C([0,T];L^2(\Omega))} + \|D_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \leqslant C\varepsilon^{\frac{1}{2}} \|u^0\|_{H_0^1(\Omega)}. \tag{1}$$

j) *Il existe une constante positive C telle que pour tous $u^0 \in L^2(\Omega)$, et $\varepsilon \in (0,1)$, on ait* :

$$\begin{aligned} &\big\|(T-t)D_\varepsilon\big\|_{C([0,T];L^2(\Omega))} + \big\|(T-t)D_\varepsilon\big\|_{L^2(0,T;H_0^1(\Omega))} \leqslant C\varepsilon^{\frac{1}{3}} \|u^0\|_{L^2(\Omega)}, \\ &\|D_\varepsilon\|_{L^2(Q)} \leqslant C\varepsilon^{\frac{1}{3}} \|u^0\|_{L^2(\Omega)}. \end{aligned} \tag{2}$$

Théorème 2. *Il existe une constante positive C telle que pour tous $u^0 \in L^2(\Omega)$, et $\varepsilon \in (0,1)$, on ait* :

$$\int_\Omega \big|u_\varepsilon(x,0)\big|^2 \,\mathrm{d}x \leqslant C \int_0^T\!\!\int_\omega u_\varepsilon^2 \,\mathrm{d}x\,\mathrm{d}t + C\varepsilon^{\frac{2}{3}} \int_Q |\nabla u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{3}$$

Théorème 3. *Soit ω un ouvert non vide quelconque dans Ω. Alors, pour tous $T > 0$, $\varepsilon \in (0,1)$, et $y^0 \in L^2(\Omega)$, il existe un contrôle $v_\varepsilon = v_{1\varepsilon}\chi_\omega + v_{2\varepsilon}$ avec $v_{1\varepsilon} \in L^2(0,T;L^2(\omega))$, et $v_{2\varepsilon} \in L^2(0,T;H^{-1}(\Omega))$, tel que la solution y_ε de* « (6) » *vérifie* (7).

De plus, il existe une constante positive C indépendante de $y^0 \in L^2(\Omega)$, et $\varepsilon \in (0,1)$ telle que :

$$\int_0^T\!\!\int_\omega v_{1\varepsilon}^2 \,\mathrm{d}x\,\mathrm{d}t + \varepsilon^{\frac{2}{3}} \int_Q |\nabla G v_{2\varepsilon}|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C \int_\Omega \big|y^0(x)\big|^2 \,\mathrm{d}x, \tag{4}$$

où G est l'inverse de $-\Delta$ avec les conditions aux limites de Dirichlet, et

$$v_{1\varepsilon} \to v \quad \textit{dans } L^2\big(0,T;L^2(\omega)\big)\ \textit{fort}, \qquad v_{2\varepsilon} \to 0 \quad \textit{dans } L^2\big(0,T;H^{-1}(\Omega)\big)\ \textit{fort}, \tag{5}$$

où v est le contrôle optimal du système limite.

1. Problem formulation and statements of main results

Let Ω be a bounded smooth open subset of $\mathbb{R}^N$, $N \geqslant 1$, and let T be a positive real number. Set $Y = (0,1)^N$, and $Q = \Omega \times (0,T)$. Let $\rho \in L^\infty_{\mathrm{per}}(\mathbb{R}^N)$ be 1-periodic in every direction, with: $\rho_0 \leqslant \rho(y) \leqslant \rho_1$, for almost every y in Y, where ρ_0 and ρ_1 are positive constants. Set $\bar\rho = \int_Y \rho(y)\,\mathrm{d}y$. Let $\varepsilon \in (0,1)$ be a small parameter. Consider the following controllability problem: Given $y^0 \in L^2(\Omega)$, can we find a control function $v_\varepsilon \in L^2(0,T;L^2(\omega))$ such that if y_ε solves

$$\begin{cases} \rho(x/\varepsilon)y_{\varepsilon t} - \Delta y_\varepsilon = v_\varepsilon & \text{in } \Omega \times (0,T), \\ y_\varepsilon = 0 \quad \text{on } \partial\Omega \times (0,T); & y_\varepsilon(x,0) = y^0(x) \quad \text{in } \Omega, \end{cases} \tag{6}$$

where ω is an arbitrary nonempty open subset of Ω, then

$$y_\varepsilon(x,T) = 0 \quad \text{a.e.} \quad \text{in } \Omega? \tag{7}$$

If the answer to this first question is positive, the natural next question is: (χ_ω denoting the characteristic function of ω), does the sequence of controls (v_ε) converge in some reasonable topology to a control v of the limit system [2,3]:

$$\begin{cases} \bar\rho y_t - \Delta y = v\chi_\omega & \text{in } \Omega \times (0,T), \\ y = 0 \quad \text{on } \partial\Omega \times (0,T); & y(x,0) = y^0(x) \quad \text{in } \Omega? \end{cases} \tag{8}$$

Concerning the first question, the use of a duality argument shows that solving that controllability problem amounts to proving the observability estimate:

$$\int_\Omega \big|u_\varepsilon(x,0)\big|^2 \,\mathrm{d}x \leqslant C_0 \int_0^T\!\!\int_\omega u_\varepsilon^2 \,\mathrm{d}x\,\mathrm{d}t \tag{9}$$

for every solution of the backward system:

$$\begin{cases} \rho(x/\varepsilon)u_{\varepsilon t} + \Delta u_\varepsilon = 0 & \text{in } \Omega \times (0,T), \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega \times (0,T); & u_\varepsilon(.,T) = u^0 \in L^2(\Omega), \end{cases} \tag{10}$$

where C_0 is a positive constant that is independent of u^0, but that may eventually depend on ε.

Since we are interested in letting ε go to zero, we want to ensure that either C_0 is independent of ε, or else the dependence with respect to ε is explicitly known. If the function ρ is smooth enough, then using Carleman estimates (e.g. [5,6]), one can show that estimate (9) holds, and the dependence of C_0 with respect to ε may be made explicit. However, the estimate obtained in this way is worthless, because the sequence of controls will diverge exponentially in $L^2(0,T;L^2(\omega))$. It then makes sense to commend the earlier work [9]; in fact, in the one-dimensional setting, its authors were able to establish a uniform boundary controllability result; their method consists in: differentiating low frequencies from high frequencies, use biorthogonal series method [4] to establish a uniform observability estimate for low frequencies, and utilize Carleman estimate for high frequencies, then use a three-step control method, which consists in dividing the time interval in three subintervals, and apply an appropriate control in each subinterval. The method of [9] that we have just roughly described is essentially one-dimensional. So is the method devised in [1] to solve the null controllability problem for parabolic equations with only bounded measurable coefficients; the method of [1] if applied to parabolic equations with fast oscillating coefficients improve the result of [9].

The methods used in [1,9] critically rely on some special changes of variables that hold in the one-dimensional setting only. Therefore a completely new approach has to be built up to prove a uniform controllability result for higher dimensional heat equation with fast oscillating density. To cope with this situation, we are going to essentially rely on the observability estimate for the limit equation:

$$\begin{cases} \bar\rho u_t + \Delta u = 0 & \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T); & u(.,T) = u^0 \in L^2(\Omega). \end{cases} \tag{11}$$

At this point, the natural question that comes to mind is: how is the use of the observability inequality for (11) going to work out to produce the uniform controllability result that we aim at? To answer this question, we observe that for some positive constant C that is independent of ε and u^0, one has:

$$\int_\Omega \left|u_\varepsilon(x,0)\right|^2 \mathrm{d}x \leqslant C \int_0^T\!\!\int_\omega u_\varepsilon^2 \,\mathrm{d}x\,\mathrm{d}t + C\int_0^T\!\!\int_\omega |u_\varepsilon - u|^2 \,\mathrm{d}x\,\mathrm{d}t + 2\int_\Omega \left|u_\varepsilon(x,0) - u(x,0)\right|^2 \mathrm{d}x. \tag{12}$$

Consequently, if one can precisely estimate the rate of convergence of u_ε to u in terms of ε and u^0, then one may derive a uniform observability estimate, that is weaker than (9) though. For the sequel we set

$$D_\varepsilon = u_\varepsilon - u.$$

Our main results read:

Theorem 1.1. i) *There exists a positive constant C such that for every $u^0 \in H_0^1(\Omega)$, and $\varepsilon \in (0,1)$, one has*:

$$\|D_\varepsilon\|_{C([0,T];L^2(\Omega))} + \|D_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \leqslant C\varepsilon^{\frac12}\left\|u^0\right\|_{H_0^1(\Omega)}. \tag{13}$$

j) *There exists a positive constant C such that for every $u^0 \in L^2(\Omega)$, and $\varepsilon \in (0,1)$, one has*:

$$\begin{gathered} \left\|(T-t)D_\varepsilon\right\|_{C([0,T];L^2(\Omega))} + \left\|(T-t)D_\varepsilon\right\|_{L^2(0,T;H_0^1(\Omega))} \leqslant C\varepsilon^{\frac13}\left\|u^0\right\|_{L^2(\Omega)}, \\ \|D_\varepsilon\|_{L^2(Q)} \leqslant C\varepsilon^{\frac13}\left\|u^0\right\|_{L^2(\Omega)}. \end{gathered} \tag{14}$$

Theorem 1.2. *There exists a positive constant C such that for every $u^0 \in L^2(\Omega)$, and $\varepsilon \in (0,1)$, one has*:

$$\int_\Omega \left|u_\varepsilon(x,0)\right|^2 \mathrm{d}x \leqslant C\int_0^T\!\!\int_\omega u_\varepsilon^2 \,\mathrm{d}x\,\mathrm{d}t + C\varepsilon^{\frac23}\int_Q |\nabla u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{15}$$

Theorem 1.3. *Let ω be any nonempty open subset of Ω. Then for all $T > 0$, $\varepsilon \in (0,1)$, and $y^0 \in L^2(\Omega)$, there exists a control function $v_\varepsilon = v_{1\varepsilon}\chi_\omega + v_{2\varepsilon}$ with $v_{1\varepsilon} \in L^2(0,T;L^2(\omega))$, and $v_{2\varepsilon} \in L^2(0,T;H^{-1}(\Omega))$, such that the solution y_ε of* '(6)' (*observe that χ_ω should be dropped in* (6)*, whence the quotes*) *satisfies* (7)*.*

Further, there exists a positive constant C independent of $y^0 \in L^2(\Omega)$, and $\varepsilon \in (0,1)$ such that:

$$\int_0^T\int_\omega |v_{1\varepsilon}|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^{\frac{2}{3}}\int_Q |\nabla G v_{2\varepsilon}|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C\int_\Omega \left|y^0(x)\right|^2\mathrm{d}x, \tag{16}$$

where G is the inverse of $-\Delta$ with Dirichlet boundary conditions, and

$$v_{1\varepsilon} \to v \quad \text{in } L^2\big(0,T;L^2(\omega)\big) \text{ strongly}, \qquad v_{2\varepsilon} \to 0 \quad \text{in } L^2\big(0,T;H^{-1}(\Omega)\big) \text{ strongly}, \tag{17}$$

where v is the optimal null control for the limit equation (8)*.*

Remark 1.4. It would have been great to have the utmost right term in (15) dropped; this would have led us to a uniform observability estimate in the standard form. But unfortunately, we were not successful in ridding ourselves of that additional term; this is the reason why an extra, and fortunately vanishing, control is needed in our approach. We also note that the structure of the control v_ε is not the one shown in (6); remember that we were looking for a control located in ω, but we found a control that breaks down into two compounds; namely $v_{1\varepsilon}$ that has the desired location, and $v_{2\varepsilon}$ that is distributed everywhere in the domain, and which vanishes as ε goes to zero.

Remark 1.5. The regularizing property of the heat equation is exhibited in estimate (14). In fact a similar estimate does not hold for the wave equation for which the state at all time and the initial data have the same smoothness. So our method will fail if attempted for the wave equation with fast oscillating density, or any other evolution system with rapidly oscillating coefficients and no smoothing property. Thus for that class of equations, we have to content ourselves, at the present time, with the uniform observability result at low frequency [7]. However, the technique to be sketched below would succeed for perturbed equations that do exhibit smoothing property; more on this in our upcoming paper [11]. We should also note that the property of establishing a uniform controllability result by adding an extra vanishing control is now well known for discretized systems (see e.g. [14]).

Before proceeding to sketching the proofs of the stated theorems, we want to specify that Theorem 1.2 being a straightforward consequence of Theorem 1.1, j), and (12), its proof will be omitted. From now on, C denotes various positive constants that may eventually depend on T, ρ, ω, and Ω, but not on ε or the initial data involved, and we set $\rho^\varepsilon(x) = \rho(x/\varepsilon)$.

2. Sketch of the proof of Theorem 1.1

The proof of Theorem 1.1 critically relies on the following result:

Lemma 2.1. 1) *(See [10].) There exists a positive constant C such that for all $w, z \in H^1(\Omega)$, one has*:

$$\left|\int_\Omega \big(\bar\rho - \rho(x/\varepsilon)\big)(wz)(x)\,\mathrm{d}x\right| \leqslant C\varepsilon\|w\|_{H^1(\Omega)}\|z\|_{H^1(\Omega)}. \tag{18}$$

2) *Let $u^0 \in L^2(\Omega)$. There exists $u^0_\varepsilon \in H^2(\Omega)\cap H^1_0(\Omega)$ such that*

$$\begin{aligned}
&\left\|u^0_\varepsilon\right\|_{H^1_0(\Omega)} \leqslant C\varepsilon^{-\frac{1}{3}}\left\|u^0\right\|_{L^2(\Omega)}, \qquad \left\|u^0_\varepsilon\right\|_{H^2(\Omega)} \leqslant C\varepsilon^{-\frac{2}{3}}\left\|u^0\right\|_{L^2(\Omega)},\\
&\left\|u^0_\varepsilon - u^0\right\|_{H^{-1}(\Omega)} \leqslant C\varepsilon^{\frac{1}{3}}\left\|u^0\right\|_{L^2(\Omega)}, \qquad u^0_\varepsilon \to u^0 \quad \text{in } L^2(\Omega) \text{ strongly}.
\end{aligned} \tag{19}$$

3) *Let $u^0 \in H^1_0(\Omega)$. There exists $w^0_\varepsilon \in H^2(\Omega)\cap H^1_0(\Omega)$ such that*

$$\begin{aligned}
&\left\|w^0_\varepsilon\right\|_{H^2(\Omega)} \leqslant C\varepsilon^{-\frac{1}{2}}\left\|u^0\right\|_{H^1_0(\Omega)}, \qquad w^0_\varepsilon \to u^0 \quad \text{in } H^1_0(\Omega) \text{ strongly},\\
&\left\|w^0_\varepsilon - u^0\right\|_{L^2(\Omega)} \leqslant C\varepsilon^{\frac{1}{2}}\left\|u^0\right\|_{H^1_0(\Omega)}.
\end{aligned} \tag{20}$$

Assertion 1) of Lemma 2.1 follows from [10, Lemma 1.6, p. 8]. To prove the assertions 2) and 3) of Lemma 1.6, one introduces appropriate perturbed elliptic problems in the spirit of Lions work on singular perturbations [8, Chap. 2].

To prove Theorem 1.1, i), one introduces the functions w_ε, and z_ε solutions of (10) and (11) respectively, with $w_\varepsilon(T) = w_\varepsilon^0$ and $z_\varepsilon(T) = w_\varepsilon^0$. Then one sets $R_\varepsilon = u_\varepsilon - w_\varepsilon$, $S_\varepsilon = w_\varepsilon - z_\varepsilon$, and $K_\varepsilon = z_\varepsilon - u$, and write down the heat equation satisfied by each of those functions. It follows from Lemma 1.6, 1) and 3), and the energy method that each of the functions R_ε, S_ε, and K_ε satisfies (13); from which we derive that $D_\varepsilon = u_\varepsilon - u$ also satisfies (13). To prove the lower inequality of (14), we proceed exactly the same way, but now using Lemma 1.6, 1) and 2), while the proof of the upper inequality relies on the lower inequality, and the smoothing property of the heat equation. □

3. Sketch of the proof of Theorem 1.3

Introduce the functional

$$\mathcal{J}_\varepsilon : L^2(\Omega) \longrightarrow \mathbb{R},$$
$$u^0 \mapsto \mathcal{J}_\varepsilon\big(u^0\big) = \frac{1}{2}\int_0^T\!\!\int_\omega \big|u_\varepsilon(x,t)\big|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon^{\frac{2}{3}}}{2}\int_Q |\nabla u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega y^0(x)u_\varepsilon(x,0)\,\mathrm{d}x, \tag{21}$$

where u_ε is the solution of (10) associated with u^0. Thanks to Theorem 1.2, one can show that $\mathcal{J}_\varepsilon$ is coercive. Further, $\mathcal{J}_\varepsilon$ is strictly convex, and continuous. Therefore, $\mathcal{J}_\varepsilon$ has a unique minimizer $\hat{u}_\varepsilon^0$, and if $\hat{u}_\varepsilon$ is the associated solution of (10), then we have the Euler equation:

$$\int_0^T\!\!\int_\omega \hat{u}_\varepsilon(x,t)u(x,t)\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^{\frac{2}{3}}\int_Q \nabla\hat{u}_\varepsilon(x,t)\cdot\nabla u(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega y^0(x)u(x,0)\,\mathrm{d}x = 0, \tag{22}$$

for every u solution of (10).

Choosing the control v_ε by: $v_\varepsilon = \hat{u}_\varepsilon\chi_\omega - \varepsilon^{\frac{2}{3}}\Delta\hat{u}_\varepsilon$, it is fairly simple to check, thanks to (22), that y_ε, the corresponding solution of (6) satisfies (7); so v_ε is a null control for (6). Using (22), and Theorem 1.2, one derives (16). Adapting the arguments develop in [9,12], one proves the claimed convergence results. □

Final Remark. Since the integral $\int_Q |\nabla u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t$ is equivalent to the $L^2(\Omega)$-norm of the initial data in (10); our approach shows how to choose the functional to be minimized (in the approximate controllability problem) in order to pass to the limit with ε in the uniform approximate controllability result of [13] to get a null (instead of an approximate) controllability result for the homogenized equation; thus with our method, we can improve the result of [13].

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