Uniform bound and a non-existence result for Lichnerowicz equation in the whole $n$-space

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Abstract

In this Note, we give a uniform bound and a non-existence result for positive solutions to the Lichnerowicz equation in $\mathbb{R}^n$. In particular, we show that positive smooth solutions to:

$$\Delta u + f(u) = 0, \quad u > 0, \quad \text{in } \mathbb{R}^n$$

where

$$f(u) = u^{p-1} - u^{p-1},$$

are uniformly bounded. To cite this article: L. Ma, X. Xu, C. R. Acad. Sci. Paris, Ser. I 347 (2009).

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1. Introduction

In the Einstein-scalar field theory one has the Lichnerowicz equation on a Riemannian manifold \((M, \gamma)\) of dimension \(n \geq 3\) (see [2,3,6]). The aim of this paper is to give some results for positive solutions to this equation in the whole Euclidean space.

Given a smooth symmetric 2-tensor \(\sigma\), a smooth vector field \(W\), and a triple data \((\pi, \tau, \varphi)\) of smooth functions on \(M\). Set

\[
c_n = \frac{n - 2}{4(n - 1)}, \quad p = \frac{2n}{n - 2},
\]

and let

\[
R_{\gamma, \varphi} = c_n \left( R(\gamma) - |\nabla \varphi|^2_\gamma \right), \quad A_{\gamma, W, \pi} = c_n (|\sigma + DW|^2_\gamma + \pi^2)
\]

and

\[
B_{\tau, \varphi} = c_n \left( \frac{n - 1}{n} \tau^2_\gamma - V(\varphi) \right)
\]

where \(V : \mathbb{R} \to \mathbb{R}\) is a given smooth function and \(R(\gamma)\) is the scalar curvature function of \(\gamma\). Then the Lichnerowicz equation for the Einstein-scalar conformal data \((\gamma, \sigma, \pi, \tau, \varphi)\) with the given vector field \(W\) is

\[
\Delta_{\gamma} u - R_{\gamma, \varphi} u + A_{\gamma, W, \pi} u^{p-1} - B_{\tau, \varphi} u^{p-1} = 0, \quad u > 0,
\]

where \(\Delta_{\gamma}\) is the Laplacian operator of \(\gamma\). We use the convention that \(\Delta_{\gamma} u = u''\) on the real line \(\mathbb{R}\). Note that \(A_{\gamma, W, \pi} \geq 0\). This equation is closely related to the Yamabe problem and the prescribing scalar curvature problems (see [1,7,8]).

We shall consider a special case when \((M, \gamma) = \mathbb{R}^n\) is the standard Euclidean space with radial symmetry data \((\sigma, \pi, \tau, \varphi)\). In this case, we can simply rewrite the equation in the following form

\[
\Delta u + R(x)u + A(x)u^{p-1} + B(x)u^{p-1} = 0, \quad u > 0, \quad \text{on } \mathbb{R}^n
\]

where \(R(x) \geq 0, A(x) \geq 0, \) and \(B(x)\) are given smooth functions of \(x \in \mathbb{R}^n\).

**Theorem 1.** Suppose that \(A := A(x) \geq 0, B := B(x) \geq 0, \) and \(R(x) \geq 0\). Let \(\beta = \frac{p + 1}{2p}\). Assume that

\[
\int_0^{+\infty} dr \left( r^{1-n} \int_{B_r(0)} A^{1-\beta} B^\beta \, dx \right) = +\infty.
\]

Then there exists no positive solution to (2).

Note that \(\beta = \frac{3p - 2}{4n}\), so the condition (3) can be written as

\[
\int_0^{+\infty} dr \left( r^{1-n} \int_{B_r(0)} A(x)^{\frac{n+2}{4n}} B(x)^{\frac{3p - 2}{4n}} \, dx \right) = +\infty.
\]

As a particular example, we note that when \(A^{1-\beta} B^\beta \geq C > 0\) for some positive constant \(C > 0\), there exists no positive solution to (2).

This result may be extended to other case (see Theorem 3 in next section).

We also have the following uniform bound for any positive solution to (2).

**Proposition 2.** Assume that \(R(x) = 0\) and \(A(x) = 1\) is a positive constant and \(B(x) = -B\) is a negative constant in (2). Then any positive solution to (2) is uniformly bounded.

In a recent paper, O. Druet and E. Hebey [4] have proved a very interesting result which says that for Lichnerowicz equation on a compact Riemannian manifold, the stability holds true when the dimension \(n\) is such that \(n \leq 5\) and fails to hold in general when \(n \geq 6\).
2. Non-existence results

In this section we prove Theorem 1. Recall our assumption that $B(x) \geq 0$ and $R(r) \geq 0$. We remark that for each fixed $x \in \mathbb{R}^n$,

$$A(x)X^{-p-1} + B(x)X^{p-1}$$

is a convex function in $X$.

Proof of Theorem 1. Let $\bar{u} := \bar{u}(r)$ be the average of $u(x)$ on the sphere $S^{n-1}_r(0)$ of radius $r$.

Note that taking this average operation and using Jensen’s inequality to Eq. (2) we have

$$-\bar{u}'' - \frac{n-1}{r} \bar{u}' \geq \frac{R(x)u}{u-1} + \frac{A(x)u^{p-1} + B(x)u^{p-1}}{u}.$$  \hspace{1cm} (4)

Using the Holder inequality to the right side of (4), we have

$$A(x)u^{p-1} + B(x)u^{p-1} \geq A_{1-\beta} B_{\beta}$$

where

$$\beta = \frac{p+1}{2p}.$$  \hspace{1cm}

Then we have

$$-r^{n-1}\bar{u}' \geq \int_{B_r(0)} A^{1-\beta} B_{\beta} \, dx + \int_{B_r(0)} R u \, dx$$

after an integration. Dividing both side by $r^{n-1}$ and integrating this inequality over $[0, r_0]$, we have

$$\bar{u}(0) - \bar{u}(r_0) \geq \frac{r_0}{\int_{B_r(0)} A^{1-\beta} B_{\beta} \, dx} + \frac{r_0}{\int_{B_r(0)} R u \, dx}.$$  \hspace{1cm}

Sending $r_0 \to \infty$ we have

$$\bar{u}(0) \geq \int_0^{\infty} \frac{r^{1-n}}{\int_{B_r(0)} A^{1-\beta} B_{\beta} \, dx} d\tau,$$

which is impossible by our assumption that

$$\int_0^{\infty} \frac{r^{1-n}}{\int_{B_r(0)} A^{1-\beta} B_{\beta} \, dx} d\tau = +\infty.$$  \hspace{1cm}

Then the proof of Theorem 1 is complete.

We remark that from our proof above, we use the interaction between $A$ and $B$. If we use the interaction between $R$ and $A$, we can have the following result by the same argument.

Theorem 3. Suppose that $A := A(x) \geq 0$, $B(x) \geq 0$, and $R := R(x) \geq 0$. Let $\beta = \frac{p+1}{2p}$. Assume that

$$\int_0^{+\infty} \frac{r^{1-n}}{\int_{B_r(0)} A^{\frac{n-2}{4(n-1)}} R(x)^{\frac{3n-2}{4(n-1)}} \, dx} d\tau = +\infty.$$  \hspace{1cm}

Then there exists no positive solution to (2).
3. Proof of Proposition 2

In this section, we assume that $R(r) = 0$ and $A(r) = 1$ is a positive constant and $B(r) = -B$ is a negative constant in (2). Then we may reduce (2) into the following form:

$$\Delta u + f(u) = 0, \quad u > 0, \quad \text{on } \mathbb{R}^n \tag{5}$$

where

$$f(u) = u^{-p-1} - Bu^{p-1}.$$  

Denote by $B_R$ any ball of radius $R > 0$ in $\mathbb{R}^n$.

We shall use a trick used in [5]. We look for a positive radial super-solution $v = v(r)$ to (5) in the ball $B_R$ with the positive infinity boundary condition. This is equivalent to finding $v = v(r) > 0$ such that

$$\begin{cases}
\Delta v + f(v) \leq 0, & \text{in } B_R, \\
v = +\infty, & \text{on } \partial B_R.
\end{cases}$$

Note that

$$f' = -(p + 1)u^{-p} - B(p - 1)u^{p-2} < 0$$

for $u > 0$. Then the comparison lemma is true for (5) in the ball $B_R$. Hence, we have

$$u(x) \leq v(r), \quad \text{in } B_R.$$  

From this we know that $u$ is uniformly bounded in $\mathbb{R}^n$.

Let $v(r) = (R^2 - r^2)^{-\alpha}$ for large $\alpha > 1$ and small $R \ll 1$. By direct computation, we know that $v$ is the right super-solution $v = v(r)$ to (5) in the ball $B_R$ with positive infinity boundary condition. Hence

$$u(x) \leq 2^\alpha R^{-2\alpha}, \quad \text{in } B_{R/2}.$$  

This proves our Proposition 2.

It is clear that our argument can be generalized to treat positive solutions to the following equation:

$$\Delta u + A(x)u^{-p-1} - Bu^{p-1} = 0, \quad \text{in } \mathbb{R}^n,$$

where $A(x)$ is a smooth uniformly bounded function in $\mathbb{R}^n$. It is an open question if the Liouville type theorem is true for positive solutions to (5).

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