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Complex Analysis

# Growth spaces on circular domains: composition operators and Carleson measures

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**Abstract**

Let $\Omega \subset \mathbb{C}^n$ be a bounded, circular and strictly convex domain with the boundary of class $\mathcal{C}^2$. Denote by $\mathcal{H}ol(\Omega)$ the space of all holomorphic functions in $\Omega$. Given $g \in \mathcal{H}ol(\Omega)$ and a holomorphic mapping $\varphi : \Omega \to \Omega$, put $C_\varphi^g f = g \cdot (f \circ \varphi)$ for $f \in \mathcal{H}ol(\Omega)$. We characterize those $g$ and $\varphi$ for which $C_\varphi^g$ is a bounded or compact operator from the growth space $\mathcal{A}^{-\log}(\Omega)$ or $\mathcal{A}^{-\beta}(\Omega)$, $\beta > 0$, to the weighted Bergman space $A_\alpha^p(\Omega)$, $0 < p < \infty$, $\alpha > -1$. Also, given $0 < q < \infty$ and $\beta > 0$, we describe those positive measures $\mu$ on $\Omega$ for which $\mathcal{A}^{-\beta}(\Omega) \subset L^q(\Omega, \mu)$ and those $\mu$ for which $\mathcal{A}^{-\log}(\Omega) \subset L^q(\Omega, \mu)$. ***To cite this article: E. Doubtsov, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***


**Résumé**

**Espaces à croissance sur les domaines circulaires : opérateurs de composition et mesures de Carleson.** Soit $\Omega$ un domaine circulaire, strictement convexe et borné dans $\mathbb{C}^n$ dont le bord est de classe $\mathcal{C}^2$. Nous désignons par $\mathcal{H}ol(\Omega)$ l'espace des fonctions holomorphes dans $\Omega$. Soient $g \in \mathcal{H}ol(\Omega)$ et $\varphi : \Omega \to \Omega$ une transformation holomorphe. Posons $C_\varphi^g f = g \cdot (f \circ \varphi)$ pour $f \in \mathcal{H}ol(\Omega)$. Nous caractérisons les fonctions $g$ et $\varphi$ pour lesquelles $C_\varphi^g$ est un opérateur borné ou compact de l'espace à croissance $\mathcal{A}^{-\log}(\Omega)$ ou de $\mathcal{A}^{-\beta}(\Omega)$, $\beta > 0$, dans l'espace de Bergman à poids $A_\alpha^p(\Omega)$, $0 < p < \infty$, $\alpha > -1$. Nous caractérisons aussi les mesures positives $\mu$ sur $\Omega$ telles que $\mathcal{A}^{-\beta}(\Omega) \subset L^q(\Omega, \mu)$ et les mesures positives $\mu$ telles que $\mathcal{A}^{-\log}(\Omega) \subset L^q(\Omega, \mu)$ pour $0 < q < \infty$ et $\beta > 0$. ***Pour citer cet article : E. Doubtsov, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***


## 1. Introduction

Throughout this Note we assume that $n \in \mathbb{N}$ and $\Omega \subset \mathbb{C}^n$ is a bounded, circular and strictly convex domain with the boundary of class $\mathcal{C}^2$. Given $z \in \Omega$, put $r_\Omega(z) = \inf\{r > 0\colon\ z/r \in \Omega\}$. Clearly, $r_\Omega(z) < 1$ for all $z \in \Omega$. If $\Omega$ is the unit ball $B_n = \{z \in \mathbb{C}^n\colon\ |z| < 1\}$, then $r_\Omega(z) = |z|$.

Let $\mathcal{H}ol(\Omega)$ denote the space of all holomorphic functions in $\Omega$. Given $\beta > 0$, the growth space $\mathcal{A}^{-\beta}(\Omega)$ consists of those $f \in \mathcal{H}ol(\Omega)$ for which $\|f\|_{-\beta} = \sup_{z\in\Omega} |f(z)|(1 - r_\Omega(z))^\beta < \infty$.

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The logarithmic growth space $\mathcal{A}^{-\log}(\Omega)$ consists of those $f \in \mathcal{H}ol(\Omega)$ for which

$$\|f\|_{-\log} = \sup_{z\in\Omega} \frac{|f(z)|}{\log(e/(1-r_\Omega(z)))} < \infty.$$

The spaces $\mathcal{A}^{-\beta}(\Omega)$, $\beta > 0$, and $\mathcal{A}^{-\log}(\Omega)$ with norms $\|\cdot\|_{-\beta}$ and $\|\cdot\|_{-\log}$ are Banach spaces.

Let $g \in \mathcal{H}ol(\Omega)$ and let $\varphi : \Omega \to \Omega$ be a holomorphic mapping. The weighted composition operator $C_\varphi^g : \mathcal{H}ol(\Omega) \to \mathcal{H}ol(\Omega)$ is defined by the formula $(C_\varphi^g f)(z) = g(z) f(\varphi(z))$, $z \in \Omega$. Given $X, Y \subset \mathcal{H}ol(\Omega)$, a standard problem is to describe those $g$ and $\varphi$ for which $C_\varphi^g$ maps $X$ to $Y$ (see, e.g., monograph [4], where the case $\Omega = B_n$ is considered in detail). In this paper, we assume that $X = \mathcal{A}^{-\beta}(\Omega)$, $\beta > 0$, or $X = \mathcal{A}^{-\log}(\Omega)$. For $\Omega = B_1$ and $X = \mathcal{A}^{-\log}(B_1)$, the main results of the present paper were recently obtained in [5].

### *1.1. Generalized Ryll–Wojtaszczyk polynomials*

Ryll and Wojtaszczyk [9] constructed holomorphic homogeneous polynomials which proved to be very useful (see, e.g., [8]). We apply similar polynomials.

**Theorem 1.1.** *(cf. [6, Theorem 2.6].) Given a domain $\Omega \subset \mathbb{C}^n$, there exist $\delta = \delta(\Omega) \in (0,1)$ and $J = J(\Omega) \in \mathbb{N}$ with the following property: For every $d \in \mathbb{N}$, there exist holomorphic homogeneous polynomials $W_j[d]$ of degree $d$, $1 \leqslant j \leqslant J$, such that*

$$\big\|W_j[d]\big\|_{L^\infty(\partial\Omega)} \leqslant 1 \quad \text{and} \quad \max_{1\leqslant j\leqslant J}\big|W_j[d](\zeta)\big| \geqslant \delta \quad \text{for all } \zeta \in \partial\Omega.$$

To prove Theorem 1.1, it suffices to repeat *mutatis mutandis* the argument used in the proof of [6, Theorem 2.6]. Remark that for $\Omega = B_n$, Theorem 1.1 was earlier proved by Aleksandrov [1]. In fact, generalizations of the Ryll–Wojtaszczyk theorem were used to obtain various results of the function theory in the unit ball of $\mathbb{C}^n$ (see, e.g., [1,10,11,2,6]). However, as far as the author is concerned, [2] is the only paper where the Ryll–Wojtaszczyk idea is applied to the study of composition operators in several complex variables.

The proof of the following key lemma is based on Theorem 1.1 and uses some ideas from [7, Proposition 5.4], where $\Omega = B_1$:

**Lemma 1.2.** *Given an $\Omega \subset \mathbb{C}^n$, there exists $M = M(\Omega) \in \mathbb{N}$ such that the following properties hold:*

1. *Let $\beta > 0$. Then there exist functions $f_m \in \mathcal{A}^{-\beta}(\Omega)$, $0 \leqslant m \leqslant M$, such that*

$$\sum_{m=0}^{M} \big|f_m(z)\big| \geqslant \frac{1}{(1-r_\Omega(z))^\beta}, \quad z \in \Omega. \tag{1}$$

2. *There exist functions $h_m \in \mathcal{A}^{-\log}(\Omega)$, $0 \leqslant m \leqslant M$, such that*

$$\sum_{m=0}^{M} \big|h_m(z)\big| \geqslant \log\frac{e}{1-r_\Omega(z)}, \quad z \in \Omega.$$

## 2. Weighted composition operators and Carleson measures

### *2.1. Composition operators from X to a lattice*

Let $\mathcal{Y}(\Omega)$ be a linear space which consists of functions $f : \Omega \to \mathbb{C}$. We say that $\mathcal{Y}(\Omega)$ is a lattice if the following property holds: if $F \in \mathcal{Y}(\Omega)$, $f \in C(\Omega)$ and $|f(z)| \leqslant |F(z)|$ for all $z \in \Omega$, then $f \in \mathcal{Y}(\Omega)$.

**Theorem 2.1.** *Assume that $g \in \mathcal{H}ol(\Omega)$, $\varphi : \Omega \to \Omega$ is a holomorphic mapping and $\mathcal{Y}(\Omega)$ is a lattice.*

1. *Let* $\beta > 0$. *The operator* $C_\varphi^g$ *maps* $\mathcal{A}^{-\beta}(\Omega)$ *to* $\mathcal{Y}(\Omega)$ *if and only if*

$$\left|g(z)\right|\bigl(1 - r_\Omega\bigl(\varphi(z)\bigr)\bigr)^{-\beta} \in \mathcal{Y}(\Omega). \tag{2}$$

2. *The operator* $C_\varphi^g$ *maps* $\mathcal{A}^{-\log}(\Omega)$ *to* $\mathcal{Y}(\Omega)$ *if and only if* $|g(z)|\log(e/(1 - r_\Omega(\varphi(z)))) \in \mathcal{Y}(\Omega)$.

**Proof.** Assume that $\beta > 0$ and $C_\varphi^g$ maps $\mathcal{A}^{-\beta}(\Omega)$ to $\mathcal{Y}(\Omega)$. Let the number $M = M(\Omega)$ and the functions $f_m \in \mathcal{A}^{-\beta}(\Omega)$, $0 \leqslant m \leqslant M$, be those provided by Lemma 1.2. Applying inequality (1), we have

$$\frac{|g(z)|}{(1 - r_\Omega(\varphi(z)))^\beta} \leqslant \sum_{m=0}^{M} \left|g(z)\right|\left|f_m\bigl(\varphi(z)\bigr)\right| = \sum_{m=0}^{M} \left|\bigl(C_\varphi^g f_m\bigr)(z)\right|, \quad z \in \Omega.$$

Since $\mathcal{Y}(\Omega)$ is a lattice, we obtain (2). To prove the converse implication, suppose that (2) holds. If $f \in \mathcal{A}^{-\beta}(\Omega)$, then $|(C_\varphi^g f)(z)| \leqslant \|f\|_{-\beta}|g(z)|(1 - r_\Omega(\varphi(z)))^{-\beta} \in \mathcal{Y}(\Omega)$. Hence, $C_\varphi^g f \in \mathcal{Y}(\Omega)$. The proof of (i) is complete. The proof of (ii) is analogous. □

*2.2. Compact operators*

The following compactness criterion is well-known:

**Lemma 2.2.** *Let* $X = \mathcal{A}^{-\beta}(\Omega)$, $\beta > 0$, *or* $X = \mathcal{A}^{-\log}(\Omega)$ *and let* $Y$ *be a linear metric space with translation invariant metric. Consider a linear operator* $T : X \to Y$. *Then the following implication holds*: *Assume that* $\{Th_j\}$ *converges to zero in the metric of* $Y$ *for any bounded in* $X$ *sequence* $\{h_j\}$ *such that* $h_j \to 0$ *uniformly on compact subsets of* $\Omega$. *Then* $T$ *is a compact operator.*

As an illustration, consider the case $Y = A_\alpha^p(\Omega)$, $0 < p < \infty$, $\alpha > -1$. The weighted Bergman space $A_\alpha^p(\Omega)$ is defined by the identity $A_\alpha^p(\Omega) = \mathcal{H}ol(\Omega) \cap L_\alpha^p(\Omega)$, where $L_\alpha^p(\Omega) = L^p(\Omega, (1 - r_\Omega(z))^\alpha\, \mathrm{d}\nu_n(z))$ and $\nu_n$ denotes Lebesgue measure on $\mathbb{C}^n$. If $1 \leqslant p < \infty$, then $A_\alpha^p(\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_{L_\alpha^p(\Omega)}$; if $0 < p < 1$, then the space $A_\alpha^p(\Omega)$ is complete with respect to the metric $d(f, g) = \|f - g\|_{L_\alpha^p(\Omega)}^p$.

**Corollary 2.3.** *Assume that* $g \in \mathcal{H}ol(\Omega)$, $\varphi : \Omega \to \Omega$ *is a holomorphic mapping,* $0 < p < \infty$ *and* $\alpha > -1$.

1. *Let* $\beta > 0$. *The operator* $C_\varphi^g$ *maps* $\mathcal{A}^{-\beta}(\Omega)$ *to* $A_\alpha^p(\Omega)$ *if and only if* $C_\varphi^g : \mathcal{A}^{-\beta}(\Omega) \to A_\alpha^p(\Omega)$ *is a compact operator if and only if*

$$\int_\Omega \frac{|g(z)|^p (1 - r_\Omega(z))^\alpha\, \mathrm{d}\nu_n(z)}{(1 - r_\Omega(\varphi(z)))^{\beta p}} < B < \infty. \tag{3}$$

2. *The operator* $C_\varphi^g$ *maps* $\mathcal{A}^{-\log}(\Omega)$ *to* $A_\alpha^p(\Omega)$ *if and only if* $C_\varphi^g : \mathcal{A}^{-\log}(\Omega) \to A_\alpha^p(\Omega)$ *is a compact operator if and only if* $\int_\Omega (|g(z)|\log(e/(1 - r_\Omega(\varphi(z)))))^p (1 - |z|)^\alpha\, \mathrm{d}\nu_n(z) < \infty$.

**Proof.** Let (3) hold. Remark that the hypotheses of Lemma 2.2 are fulfilled for $T = C_\varphi^g : \mathcal{A}^{-\beta}(\Omega) \to A_\alpha^p(\Omega)$. So, assume that $h_j \in \mathcal{A}^{-\beta}(\Omega)$, $\|h_j\|_{-\beta}^p < H < \infty$ and $h_j \to 0$ uniformly on compact subsets of $\Omega$.

Fix an $\varepsilon > 0$. By (3), if a compact $K_0 \subset \Omega$ is large enough, then

$$\int_{\Omega \setminus K_0} \frac{|g(z)|^p (1 - r_\Omega(z))^\alpha\, \mathrm{d}\nu_n(z)}{(1 - r_\Omega(\varphi(z)))^{\beta p}} < \frac{\varepsilon}{2H}.$$

Put $K = \varphi(K_0)$, then $K$ is a compact subset of $\Omega$ and $\varphi^{-1}(\Omega \setminus K) \subset \Omega \setminus K_0$. Hence,

$$\int_{\varphi^{-1}(\Omega \setminus K)} \left|\bigl(C_\varphi^g h_j\bigr)(z)\right|^p \bigl(1 - r_\Omega(z)\bigr)^\alpha\, \mathrm{d}\nu_n(z) < H \int_{\Omega \setminus K_0} \frac{|g(z)|^p (1 - r_\Omega(z))^\alpha\, \mathrm{d}\nu_n(z)}{(1 - r_\Omega(\varphi(z)))^{\beta p}} < \frac{\varepsilon}{2}$$

for all $j$. By assumption, $|h_j(w)|^p < \frac{\varepsilon}{2B}$ for all $w \in K$, $j \geqslant j_0$. Hence,

$$\int\limits_{\varphi^{-1}(K)} \left|\left(C_\varphi^g h_j\right)(z)\right|^p \left(1 - r_\Omega(z)\right)^\alpha \,\mathrm{d}\nu_n(z) < \frac{\varepsilon}{2B} \int\limits_{\Omega} \left|g(z)\right|^p \left(1 - r_\Omega(z)\right)^\alpha \,\mathrm{d}\nu_n(z) < \frac{\varepsilon}{2}$$

for all $j \geqslant j_0$. So $\|C_\varphi^g h_j\|_{A_\alpha^p}^p < \varepsilon$ for all $j \geqslant j_0$. Therefore, Lemma 2.2 guarantees that $C_\varphi^g$ is a compact operator from $\mathcal{A}^{-\beta}(\Omega)$ to $A_\alpha^p(\Omega)$. Remark that $L_\alpha^p(\Omega)$ is a lattice, thus, by Theorem 2.1, the proof of part 1 is complete. The proof of part 2 is analogous. □

### 2.3. Carleson measures

Given $\mathcal{X} \subset \mathcal{H}ol(\Omega)$ and $0 < q < \infty$, a well-known problem is to characterize those positive measures $\mu$ on $\Omega$ for which $\mathcal{X} \subset L^q(\Omega, \mu)$. The corresponding measures $\mu$ are called $q$-Carleson for $\mathcal{X}$. Carleson [3] solved the problem when $\Omega$ is the unit disk $B_1$ and $\mathcal{X}$ is the Hardy space $H^q(B_1)$. By now, characterizations of the $q$-Carleson measures are known for various classical spaces $\mathcal{X}$ of holomorphic functions. The proof of the following result is similar to that of Corollary 2.3:

**Corollary 2.4.** *Let $0 < q < \infty$ and let $\mu$ be a positive measure on $\Omega$.*

1. *Let $\beta > 0$. Then $\mu$ is a $q$-Carleson measure for $\mathcal{A}^{-\beta}(\Omega)$ if and only if the identity operator $I : \mathcal{A}^{-\beta}(\Omega) \to L^q(\Omega, \mu)$ is compact if and only if $\int_\Omega (1 - r_\Omega(z))^{-\beta q}\,\mathrm{d}\mu(z) < \infty$.*
2. *The measure $\mu$ is $q$-Carleson for $\mathcal{A}^{-\log}(\Omega)$ if and only if the identity operator $I : \mathcal{A}^{-\log}(\Omega) \to L^q(\Omega, \mu)$ is compact if and only if $\int_\Omega (\log(e/(1 - r_\Omega(z))))^q\,\mathrm{d}\mu(z) < \infty$.*

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