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Topology

Codimension one minimal foliations and the higher homotopy groups of leaves

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Abstract

Let $\mathcal{F}$ be a codimension one foliation of an aspherical manifold M. Assume that $\mathcal{F}$ has no vanishing cycles. If there is an aspherical dense leaf of $\mathcal{F}$, then each leaf of $\mathcal{F}$ is aspherical. If $\mathcal{F}$ is minimal and the universal covering of a leaf of $\mathcal{F}$ is not k-connected, then the universal coverings of no leaves are k-connected. ***To cite this article: T. Yokoyama, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Feuilletages de codimension un et des groupes d'homotopie d'ordre supérieur de feuilles. Soit $\mathcal{F}$ une feuilletage de codimension un sur une variété M asphérique. Supposons que $\mathcal{F}$ n'a pas de cycles évanouissants. S'il y a une feuille asphérique et dense, alors toute feuille de $\mathcal{F}$ est asphérique. Si $\widetilde{\mathcal{F}}$ est minimal et le revêtement universel d'une feuille n'est pas k-connexe, alors le revêtement universel d'aucune feuille est k-connexe. ***Pour citer cet article : T. Yokoyama, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Let $\mathcal{F}$ be a codimension one foliation of a paracompact manifold M. We are interested in the relationship between the topology of the leaves of $\mathcal{F}$ and the topology of M, in particular, the asphericity of the manifold and that of leaves.

By the result of Lamoureux [1], if $H_k M$ has a nontrivial spherical element, then there is a leaf $L \in \mathcal{F}$ such that $\pi_1 L$, $H_{k-1} L$, or $H_k L$ is nontrivial. By considering the induced foliation $\mathcal{F}$ of the universal covering $\widetilde{M}$, this result implies the following proposition:

Proposition 1.1. *(See [1].) If M admits an aspherical foliation (i.e. a foliation of which all leaves are aspherical) without vanishing cycles, then M is aspherical.*

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Here a loop γ on the leaf $L \in \mathcal{F}$ is called a vanishing cycle if there is a mapping $F : S^1 \times [0,1] \to M$ such that curves $F(x,[0,1])$ for $x \in S^1$ are transverse to $\mathcal{F}$, each loop $F(S^1,t)$ for $t \in [0,1]$ is contained in a leaf L_t, $\gamma = F(S^1,0)$ is nontrivial in $\pi_1 L$ and $F(S^1,t)$ is trivial in $\pi_1 L_t$ for $t \in (0,1]$.

If the foliation $\mathcal{F}$ of a manifold M is without vanishing cycles, then the induced foliation $\widetilde{\mathcal{F}}$ of the universal covering $\widetilde{M}$ of M consists of simply connected leaves and is transversely orientable.

Note that the Reeb foliation of S^3 is an aspherical foliation with nontrivial vanishing cycles. It is not difficult to construct minimal aspherical foliations with nontrivial vanishing cycles of non-compact aspherical manifolds.

We would like to know whether the asphericity of M implies the asphericity of the leaves of $\mathcal{F}$. For a 3-dimensional aspherical manifold M^3, any foliation $\mathcal{F}$ of M^3 is aspherical. For $n \geqslant 4$, there are non-aspherical foliations of n-dimensional aspherical manifolds which are obtained from given foliations by the turbulization along a closed transverse curve. So our question is to know whether minimal foliations or foliations with dense leaves of closed aspherical manifolds are always aspherical.

In this Note, we show the asphericity of foliations under some additional assumptions.

A topological space X is said to be exactly k-connected if X is k-connected but not $(k+1)$-connected, and X is said to be (resp. exactly) k-aspherical if the universal covering of X is (resp. exactly) k-connected.

Theorem 1.2. *Let $\mathcal{F}$ be a codimension one foliation without vanishing cycles of a k-aspherical manifold M, where $k \geqslant 3$. If there is a dense leaf of $\mathcal{F}$ which is $(k-1)$-aspherical, then each leaf of $\mathcal{F}$ is $(k-1)$-aspherical. In particular, for an aspherical manifold M if there is an aspherical dense leaf of $\mathcal{F}$, then each leaf of $\mathcal{F}$ is aspherical.*

Theorem 1.3. *Let $\mathcal{F}$ be a codimension one minimal foliation without vanishing cycles of a manifold M. Suppose that M is $(k+1)$-aspherical. If there is a leaf of $\mathcal{F}$ is exactly $(k-1)$-aspherical, then each leaf of $\mathcal{F}$ is exactly $(k-1)$-aspherical.*

To clarify the meaning of Theorem 1.3, we will give an example of minimal foliation of an exactly $(k-1)$-aspherical manifold M with exactly $(k-2)$-aspherical leaves and exactly $(k-1)$-aspherical leaves.

2. Proofs

Let $\widetilde{\mathcal{F}}$ be a codimension one transversely oriented foliation of a simply connected manifold X which consists of simply connected closed leaves. Let $\widetilde{L}_0$ be a leaf of $\widetilde{\mathcal{F}}$ such that $H_i \widetilde{L}_0 \neq 0$ for $(i \geqslant 2)$. Then there is a compact connected subset K such that $j_*(H_i K)$ is nontrivial in $H_i \widetilde{L}_0$ where $j : K \to \widetilde{L}_0$ is the inclusion. Since $\widetilde{L}_0$ is simply connected, there is an embedding $F : K \times [-\varepsilon, \varepsilon] \to X$, called a trivial two-sided fence on K with the following properties: $F(K \times \{t\})$ is contained in a leaf L_t of $\mathcal{F}$ $(t \in [-\varepsilon,\varepsilon])$, $F|K \times \{0\}$ is the inclusion $K \subset L_0$ and $F(\{x\} \times [-\varepsilon,\varepsilon])$ is an orientation preserving embedding transverse to $\mathcal{F}$ $(x \in K)$.

Let $F_{t*} : H_i K \to H_i \widetilde{L}_t$ be the induced homomorphism.

Lemma 2.1. *If there are $-\varepsilon \leqslant t_- < 0 < t_+ \leqslant \varepsilon$ such that $F_{t_\pm *}(H_i K) = 0 \leqslant H_i \widetilde{L}_{t_\pm}$, then $H_{i+1} X \neq 0$.*

Proof. Let X_-, X_+ be the closures of the connected components of the complement of $\widetilde{L}_0$. The Mayer–Vietoris exact sequence for (X_-, X_+) is as follows:

$$\cdots \longrightarrow H_{i+1} X \longrightarrow H_i \widetilde{L}_0 \xrightarrow{f} H_i X_- \oplus H_i X_+ \longrightarrow \cdots.$$

Since $j_*(H_i K) \leqslant H_i \widetilde{L}_0$ is nontrivial, the factorization

$$f \mid j_*(H_i K) : j_*(H_i K) \to F_{t_+ *}(H_i K) \oplus F_{t_- *}(H_i K) = 0 \oplus 0 \to H_i X_- \oplus H_i X_+$$

implies that f is not injective. Hence $H_{i+1} X \to H_i \widetilde{L}_0$ is not the 0-map and $H_{i+1} X$ is nontrivial. □

Proof of Theorem 1.2. Let $\widetilde{M}$ be the universal covering of M and $\widetilde{\mathcal{F}}$ be the pullback foliation of $\mathcal{F}$ of $\widetilde{M}$. Since there are no vanishing cycles, $\widetilde{\mathcal{F}}$ consists of simply connected closed leaves. Fix a transverse orientation of $\widetilde{\mathcal{F}}$. Let L be the dense leaf of $\mathcal{F}$ which is $(k-1)$-aspherical. If there is a leaf L' of $\mathcal{F}$ which is exactly $(i-1)$-aspherical for $1 < i \leqslant k-1$, for a lift $\widetilde{L}'$ of L', $H_i \widetilde{L}' \cong \pi_i \widetilde{L}' \neq 0$. Hence there is a compact subset $K \subset \widetilde{L}'$ such that $j_*(H_i K) \leqslant H_i \widetilde{L}'$

is non trivial, where $j: K \to \widetilde{L}'$ is the inclusion. Take a two-sided trivial fence $F: K \times [-\varepsilon, \varepsilon] \to \widetilde{M}$. Since L is dense, the image of F intersects lifts of L on both sides of $\widetilde{L}'$. Since all lifts of L have trivial H_i, the trivial fence F satisfies the conditions in Lemma 2.1. By Lemma 2.1, $H_{i+1}\widetilde{M} \neq 0$. This contradicts the fact that $\widetilde{M}$ is k-connected. Thus all leaves of $\widetilde{\mathcal{F}}$ are $(k-1)$-aspherical. □

Proof of Theorem 1.3. Let $\widetilde{M}$ and $\widetilde{\mathcal{F}}$ as above. Let $\widetilde{L}'$ be a leaf of $\widetilde{\mathcal{F}}$ which is exactly $(k-1)$-connected. By Theorem 1.2, all leaves of $\widetilde{\mathcal{F}}$ are $(k-1)$-connected. If there is a leaf $\widetilde{L}''$ which is k-connected, then Theorem 1.2 implies that all leaves of $\widetilde{\mathcal{F}}$ are k-connected. This contradicts the assumption. Hence all leaves are exactly $(k-1)$-connected. □

3. An example

Consider $M_1 = ((D^k \times S^1) - \operatorname{int} D^{k+1}) \times S^1$ ($k \geqslant 3$) and a transversely oriented foliation on M_1 positively tangent to the boundary component $L_1 = S^{k-1} \times S^1 \times S^1$ and negatively tangent to the boundary component $L_2 \cong S^k \times S^1$ such that the restriction to the interior $\operatorname{int}((D^k \times S^1 - \operatorname{int} D^{k+1}) \times S^1)$ is isomorphic to the product foliation defined by the projection to S^1. Take a transverse curve γ joining the two boundary components. Let M_2 be the double of M_1. Then the double of γ is a closed transverse curve $\overline{\gamma}$. Let $(M', \mathcal{F}') = (S^1 \times S^1 \times S^k, \mathcal{F} \times S^k)$, where $\mathcal{F}$ is the foliation of $S^1 \times S^1$ by the lines of an irrational slope. Take the closed transverse curve $\gamma' = S^1 \times 0 \times \{x\}$ ($x \in S^k$) to $\mathcal{F}'$. Let $N' = M' - U_{\gamma'}$ and $N_2 = M_2 - U_{\overline{\gamma}}$ where $U_{\gamma'}$ and $U_{\overline{\gamma}}$ are tubular neighborhoods of γ' and $\overline{\gamma}$, respectively. Then $\partial N' \cong S^1 \times S^k$ and $\partial N_2 \cong S^1 \times S^k$ are foliated by product foliations and N_2 has two compact leaves $L_1 - \operatorname{int} D^{k+1}$ and $L_2 - \operatorname{int} D^{k+1}$. Let $(M_3, \mathcal{F}_3)$ be the foliated manifold obtain from N_2 and N' by identifying their boundaries by a foliation preserving diffeomorphism such that $L_1 - \operatorname{int} D^{k+1}$ and $L_2 - \operatorname{int} D^{k+1}$ do not belong to the same leaf.

$(M_3, \mathcal{F}_3)$ has the following properties: M_3 is exactly $(k-1)$-aspherical and $\mathcal{F}_3$ is a codimension one minimal C^∞ transversely orientable foliation without vanishing cycles which has exactly $(k-2)$-aspherical leaves and an exactly $(k-1)$-aspherical leaf.

First we show that M_3 is $(k-1)$-aspherical. M_2 is obtained from

$$A = \left(S^k \times S^1 - \left\{\operatorname{int} D_1^{k+1} \sqcup \operatorname{int} D_2^{k+1}\right\}\right) \times S^1$$

by attaching its boundary components. Hence the universal covering $\widetilde{M}_2$ of M_2 is obtained from countably many copies of

$$\widetilde{A} = \left(S^k \times \mathbb{R} - \bigsqcup_{l \in \mathbb{Z}} \left(\left(\operatorname{int} D_1^{k+1}\right)_l \sqcup \left(\operatorname{int} D_2^{k+1}\right)_l\right)\right) \times \mathbb{R}$$

attaching one boundary component $(\partial D_1^{k+1})_j \times \mathbb{R}$ to another $(\partial D_2^{k+1})_j \times \mathbb{R}$. Hence $\widetilde{M}_2$ is $(k-1)$-connected. The universal covering $\widetilde{M}_3$ of M_3 is obtained from countably many copies of

$$\widetilde{N}_2 \cong \widetilde{M}_2 - \bigsqcup_{l \in \mathbb{Z}} \left(\left(\operatorname{int} D^{k+1}\right)_l \times \mathbb{R}\right) \quad \text{and} \quad \widetilde{N}' \cong \widetilde{M}' - \bigsqcup_{\mathbb{Z}} \left(\left(\operatorname{int} D^{k+1}\right)_l \times \mathbb{R}\right)$$

by attaching one boundary component to another. Hence $\widetilde{M}_3$ is $(k-1)$-connected. We associate $\widetilde{M}_3$ with a tree G whose vertices correspond to copies of $\widetilde{N}_2$ or $\widetilde{N}'$ and whose edges correspond to the identified boundaries diffeomorphic to $S^{k+1} \times \mathbb{R}$.

Secondly the minimality of $\mathcal{F}'$ implies that of $\mathcal{F}_3$. We show that the leaves of the induced foliation $\widetilde{\mathcal{F}}_3$ of $\widetilde{M}_3$ are simply connected. Then there are no vanishing cycles.

It easy to see that the leaves of $(\widetilde{M}_2, \widetilde{\mathcal{F}}_2)$ are homeomorphic to $S^{k-1} \times \mathbb{R} \times \mathbb{R}$, $S^k \times \mathbb{R}$, or $D^k \times \mathbb{R} - \bigsqcup_{\mathbb{Z}} \operatorname{int} D^{k+1}$ and the leaves of $(\widetilde{M}', \widetilde{\mathcal{F}}')$ are homeomorphic to $\mathbb{R} \times S^k$. Note here that the intersection of a leaf of $\widetilde{N}_2$ or $\widetilde{N}'$ and one of the boundary components of $\widetilde{N}_2$ or $\widetilde{N}'$ is diffeomorphic to S^k. For each leaf $\widetilde{L}$ of $\widetilde{M}_3$, the intersection of $\widetilde{L}$ and a copy of $\widetilde{N}_2$ or $\widetilde{N}'$ is diffeomorphic to

$$S^{k-1} \times \mathbb{R} \times \mathbb{R} - \bigsqcup_{l \in \mathbb{Z}} \left(\operatorname{int} D^{k+1}\right)_l, \quad S^k \times \mathbb{R} - \bigsqcup_{l \in \mathbb{Z}} \left(\operatorname{int} D^{k+1}\right)_l, \quad \text{or} \quad \left(D^k \times \mathbb{R} - \bigsqcup_{\mathbb{Z}} \operatorname{int} D^{k+1}\right) - \bigsqcup_{l \in \mathbb{Z}} \left(\operatorname{int} D^{k+1}\right)_l.$$

Hence $\widetilde{L}$ corresponds to a subtree of the tree G. Since $\widetilde{L}$ is obtained from these copies by attaching along their boundary $(\partial D^{k+1})_j$ according to the subtree. Thus $\widetilde{L}$ is simply connected.

From the description of the leaves, the leaf containing $S^{k-1} \times S^1 \times S^1 - \operatorname{int} D^{k+1}$ is exactly $(k-2)$-aspherical and all other leaves are exactly $(k-1)$-aspherical.

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