Numerical Analysis

A residual based a posteriori estimator for the reaction-diffusion problem

Mika Juntunen, Rolf Stenberg

Helsinki University of Technology (TKK), Department of Mathematics and Systems Analysis, P. O. Box 1100, 02015 TKK, Finland

Received 22 February 2009; accepted 16 March 2009
Available online 5 April 2009
Presented by Olivier Pironneau

Abstract

A residual based a posteriori estimator for the reaction-diffusion problem is introduced. We show that the estimator gives both an upper and a lower bound to error. Numerical results are presented. To cite this article: M. Juntunen, R. Stenberg, C. R. Acad. Sci. Paris, Ser. I 347 (2009).

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1. Introduction

We consider the finite element approximation of the reaction-diffusion problem

\[-\varepsilon^2 \Delta u + u = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial \Omega,\]

with the parameter \( \varepsilon > 0 \). For \( \varepsilon \gtrsim 1 \) the problem is a standard elliptic equation. We are, however, interested in the case of a “small” \( \varepsilon \ll 1 \). In this case, the problem is a singularly perturbed problem, and the question is how to incorporate the effect of \( \varepsilon \) into the finite element a posteriori analysis. The problem has been studied for example in [4,1]. Here we introduce and analyze an alternative a posteriori estimator. In [2], this is extended to the Brinkman equations modeling flow in porous media.

E-mail addresses: mika.juntunen@tkk.fi (M. Juntunen), rolf.stenberg@tkk.fi (R. Stenberg).

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2. The a posteriori error estimate

Let $\Omega \subset \mathbb{R}^n$ be a domain with a polygonal or a polyhedral boundary $\partial \Omega$. We assume a shape regular triangular/tetrahedral partitioning $C_h$ of the domain $\Omega$. With $h_K$ we denote the diameter of $K \in C_h$ and we let $h = \max h_K$. With $\mathcal{E}_h$ we denote the internal edges (faces in 3D) of $C_h$. The constant $C$ is a generic constant independent of the mesh size and problem parameter $\varepsilon$.

Defining the bilinear form
\[ A(u, v) = \varepsilon^2 (\nabla u, \nabla v) + (u, v), \tag{2} \]
the weak form of the problem is: find $u \in V$ such that
\[ A(u, v) = (f, v) \quad \forall v \in H^1_0(\Omega). \tag{3} \]
Defining $V_h = \{ v \in H^1_0(\Omega) \mid v|_K \in P_k(K) \forall K \in C_h \}$, the finite element method is: find $u_h \in V_h$ such that
\[ A(u_h, v) = (f, v) \quad \forall v \in V_h. \tag{4} \]

The natural energy norm is
\[ \| v \|_\varepsilon^2 = \varepsilon^2 \| \nabla v \|_0^2 + \| v \|_0^2, \tag{5} \]
and the finite element solution is the best approximation with respect to this norm
\[ \| u - u_h \|_\varepsilon = \inf_{v \in V_h} \| u - v \|_\varepsilon. \tag{6} \]

In general, the problem has a boundary layer of the form $e^{-d/\varepsilon}$, where $d$ is the distance from the boundary. Hence, even for a smooth load $f$, a uniform mesh will only lead to the following estimate:
\[ \| u - u_h \|_\varepsilon \leq C \sqrt{h} \tag{7} \]
uniformly valid with respect to $\varepsilon$. For a smooth solution the estimate obtained is
\[ \| u - u_h \|_\varepsilon \leq C (\varepsilon h^k + h^{k+1}). \tag{8} \]

To improve the convergence, adaptive mesh refinement is natural. Here, we introduce a novel residual based a posteriori estimator. The elementwise estimator is defined as
\[ E_K(u_h)^2 = \frac{h_K^2}{\varepsilon^2 + h_K^2} \| \varepsilon^2 \Delta u_h - u_h + f \|_{0,K}^2 + \frac{h_K}{\varepsilon^2 + h_K^2} \| \varepsilon^2 \partial_n u_h \|_{0,\partial K \cap \mathcal{E}_h}^2 \tag{9} \]
and the global estimator is
\[ \eta = \left( \sum_{K \in C_h} E_K(u_h)^2 \right)^{1/2}. \tag{10} \]

Above $[]$ denotes the jump and $\partial_n$ denotes the normal derivative.

If $\varepsilon \gtrsim 1$, the elementwise estimator recovers the usual estimator for second order elliptic equations
\[ E_K(u_h)^2 \approx h_K^2 \| \varepsilon^2 \Delta u_h - u_h + f \|_{0,K}^2 + h_K \| \varepsilon^2 \partial_n u_h \|_{0,\partial K \cap \mathcal{E}_h}^2 \]
and on the other hand, in the limit $\varepsilon \to 0$ (or $\varepsilon \ll h$), when the FE solution is the $L^2$-projection of the loading, we have
\[ E_K(u_h)^2 \approx \| -u_h + f \|_{0,K}^2. \]

For our analysis we will need a saturation assumption. The partitioning $C_h$ is refined into $C_{h/2}$ by dividing each triangle/tetrahedron $K$ into four/eight elements with mesh size $h_K/2$. By $u_{h/2} \in V_{h/2}$ we denote the finite element solution on the refined mesh.

**Assumption 2.1.** There exists a positive constant $\beta < 1$ such that
\[ \| u - u_{h/2} \|_\varepsilon \leq \beta \| u - u_h \|_\varepsilon. \tag{11} \]
The main result is the following theorem:

**Theorem 2.2.** Let Assumption 2.1 hold. Then there exists $C > 0$ such that

$$\|u - u_h\|_e \leq C \eta. \quad (12)$$

**Proof.** By the triangle inequality the saturation assumption gives

$$\|u - u_h\|_e \leq \frac{C}{1 - \beta} (\|u_{h/2} - u_h\|_e). \quad (13)$$

Next, with $v = (u_{h/2} - u_h)/\|u_{h/2} - u_h\|_e$, we have

$$\|u_{h/2} - u_h\|_e = \mathcal{A}(u_{h/2} - u_h, v) \quad (14)$$

and $\|v\|_e = 1$. Let $\tilde{v} \in V_h$ be the Lagrange interpolant of $v$. Since both $v$ and $\tilde{v}$ are in the finite element spaces, scaling arguments give

$$\left( \sum_{K \in \mathcal{C}_{h/2}} \left( \frac{\varepsilon + h_K}{h_K} \right)^2 \|v - \tilde{v}\|_{0,K}^2 \right)^{1/2} \leq C \left( \sum_{K \in \mathcal{C}_{h/2}} \left( \frac{\varepsilon^2 \|\nabla v\|_{0,K}^2 + \|v\|_{0,K}^2}{h_K} \right) \right)^{1/2} = C \|v\|_e = C. \quad (15)$$

and

$$\left( \sum_{K \in \mathcal{C}_{h/2}} \frac{\varepsilon^2 + h_K^2}{h_K} \|v - \tilde{v}\|_{0,\partial K}^2 \right)^{1/2} \leq C \left( \sum_{K \in \mathcal{C}_{h/2}} \frac{\varepsilon^2 + h_K^2}{h_K} h_K^{-1} \|v - \tilde{v}\|_{0,K}^2 \right)^{1/2}$$

$$= C \left( \sum_{K \in \mathcal{C}_{h/2}} \left( \frac{\varepsilon^2}{h_K^2} + 1 \right) \|v - \tilde{v}\|_{0,K}^2 \right)^{1/2} \leq C \left( \sum_{K \in \mathcal{C}_{h/2}} \left( \varepsilon^2 \|\nabla v\|_{0,K}^2 + \|v\|_{0,K}^2 \right) \right)^{1/2} = C \|v\|_e = C. \quad (16)$$

Since it holds $\mathcal{A}(u_{h/2} - u_h, \tilde{v}) = 0$, we have

$$\mathcal{A}(u_{h/2} - u_h, v) = \mathcal{A}(u_{h/2} - u_h, v - \tilde{v}). \quad (17)$$

Using the fact that $u_{h/2}$ satisfies

$$\mathcal{A}(u_{h/2}, v - \tilde{v}) = (f, v - \tilde{v}) \quad (18)$$

and integrating by parts, we get

$$\mathcal{A}(u_{h/2} - u_h, v - \tilde{v}) = (f, v - \tilde{v}) - \varepsilon^2 (\nabla u_h, \nabla (v - \tilde{v})) - (u_h, v - \tilde{v})$$

$$= \sum_{K \in \mathcal{C}_{h/2}} \left\{ (\varepsilon^2 \Delta u_h - f, v - \tilde{v})_K + \varepsilon^2 (\partial_n u_h, v - \tilde{v})_{\partial K \cap \mathcal{C}_{h/2}} \right\}. \quad (19)$$

Using Schwartz inequality and the estimates (15)–(16) we then obtain

$$\mathcal{A}(u_{h/2} - u_h, v - \tilde{v}) \leq C \eta. \quad \square \quad (20)$$

The a posteriori upper bound $\eta$ is also a lower bound to the error. In this sense the estimator is sharp. The proof of the following theorem uses classical techniques, see [3]:

**Theorem 2.3.** Let $f_h \in V_h$ be an approximation of the load $f$. Then there exist $C > 0$ such that

$$\eta^2 \leq C \left\{ \|u - u_h\|_e^2 + \sum_{K \in \mathcal{C}_h} \left( \frac{h_K^2}{\varepsilon^2 + h_K^2} \|f - f_h\|_{0,K}^2 \right) \right\}. \quad (21)$$
Fig. 1. Upper panels: Convergence for uniform and adaptive meshes for parameter values $\varepsilon = 0.05$ and $\varepsilon = 0.01$. Lower panels: First three meshes of the adaptive scheme using linear elements and parameter value $\varepsilon = 0.05$.

### 3. Numerical results

For the computations we choose the unit square $\Omega = (0, 1) \times (0, 1)$ and a unit load $f = 1$. For the number of degrees of freedom $N$, the uniform estimate (7) and the asymptotic estimate (8) become

$$
\| u - u_h \|_\varepsilon \leq CN^{-0.25} \quad \text{and} \quad \| u - u_h \|_\varepsilon \leq C(\varepsilon N^{-k/2} + N^{-(k+1)/2}),
$$

respectively. In Fig. 1 this behavior is seen for linear and quadratic elements ($k = 1, 2$).

### References


