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Complex Analysis

The Plemelj–Privalov theorem in Clifford analysis

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Abstract

This Note gives geometric conditions on a surface of $\mathbb{R}^n$ so that the Hilbert transform on that surface in the framework of Clifford analysis defines a bounded operator in the Hölder continuous functions classes. This result provides a generalization of the well-known theorem of Plemelj and Privalov for curves in $\mathbb{R}^2$. ***To cite this article: R. Abreu Blaya et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Le theorème de Plemelj–Privalov au domaine de l'analyse de Clifford. Cette Note propose une condition géométrique sur une surface de $\mathbb{R}^n$ de façon que la transformée de Hilbert sur cette surface, dans le contexte de l'analyse de Clifford, définisse un opérateur borné dans les classes de fonctions de Hölder. Cet résultat généralise le théorème bien connu de Plemelj et Privalov pour des courbes de $\mathbb{R}^2$. ***Pour citer cet article : R. Abreu Blaya et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

The classical Plemelj–Privalov theorem states that if γ is a circle, then the Hilbert transform

$$f \mapsto \frac{1}{\pi i} \int_{\gamma} \frac{f(\xi) - f(t)}{\xi - t} \, \mathrm{d}\xi + f(t) \tag{1}$$

is bounded in the class of Hölder continuous functions, denoted by $C^{0,\alpha}(\gamma)$ for $0 < \alpha < 1$.

In 1975, Salaev [9] proved that the Plemelj–Privalov theorem is still true if γ is assumed to be a closed rectifiable Jordan curve, which is required to be AD-regular. The latter assumption means that there is a $c > 0$ so that for all $\xi_0 \in \mathbb{C}$ and all $r > 0$ the arc length measure of $\{\xi\colon |\xi - \xi_0| \leqslant r\} \cap \gamma$ is at most cr.

Let us remind the reader that a closely related question is to know when there is L^p-boundedness of (1). David [4] has shown that for each p, $1 < p < \infty$ the Hilbert transform is bounded on $L^p(\gamma)$ if and only if γ is AD-regular.

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In contrast to the above result, the $C^{0,\alpha}$-boundedness of (1) is not enough to imply AD-regularity of γ. In [10] it was shown that the operator (1) is bounded in $C^{0,\alpha}(\gamma)$ if and only if γ satisfies a weaker geometrical condition being denoted by $\mathcal{V}_\alpha$. For $0<\alpha<1$, the condition $\mathcal{V}_\alpha$ is satisfied by any AD-regular curve, and so by many others.

The question we shall be concerned with is whether or not similar results can be described in a higher dimensional context.

In order to present a positive answer to this question, we shall restrict ourselves to $C^{0,\alpha}$ estimates instead of L^p ones. Indeed, it is worthwhile to notice that for $(n-1)$-dimensional surfaces Γ in $\mathbb{R}^n$, $n>2$, such a simple David's characterization seems impossible. A deeper discussion of this topic can be found in [5].

Recently a version of the Plemelj–Privalov theorem in higher dimensions has been given in [2] by using the methods of Clifford analysis. For a classical account of this function theory we refer the reader to [3].

More precisely, it has been proved that the Clifford–Hilbert transform given by

$$\mathrm{H}f(x)=\int_\Gamma \frac{x-y}{\sigma_n|y-x|^n}\nu(y)\big(f(y)-f(x)\big)\,\mathrm{d}\mathcal{H}^{n-1}(y)+f(x), \tag{2}$$

is bounded in $C^{0,\alpha}(\Gamma)$ whenever Γ is an AD-regular surface. Here σ_n denotes the area of the unit sphere in $\mathbb{R}^n$.

This time $C^{0,\alpha}(\Gamma)$, equipped with the standard norm $\|\,\|_\alpha$, stands for the real Banach space of all $\mathbb{R}_{0,n}$-valued functions which are componentwise Hölder continuous on Γ and the integrand in (2) is to be interpreted in the sense of the product defined in the Clifford algebra $\mathbb{R}_{0,n}$. Here and in the sequel, $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure (see [8]) and $\nu(y)$ is the unit normal vector on Γ at the point y in the sense of Federer (see [6]).

To make our exposition self-contained, we recall that a surface Γ in $\mathbb{R}^n$ is said to be AD-regular if there exists a constant $c>0$, such that, for all $x\in\Gamma$ and $0<r\leqslant\operatorname{diam}(\Gamma)$: $c^{-1}r^{n-1}\leqslant\mathcal{H}^{n-1}(\Gamma\cap B(x,r))\leqslant cr^{n-1}$, where $B(x,r)$ is the closed ball with center x and radius r (see [5,8]).

We are interested in finding the maximal class of surfaces for which the multidimensional Plemelj–Privalov theorem remains valid. To this end, nothing is assumed concerning neither the rectifiability nor AD-regularity of Γ, but a less restrictive assumption given by condition (5) in § 1. 1 is made.

We emphasize that condition (5) was already introduced in [1] and gives rise to a very general class of surfaces that contains all classes of those classically considered in the literature.

Before stating the main result of the paper we will need some geometric preliminaries.

1.1. Some geometry

In this subsection we use and extend some of the ideas presented in [7,10] for curves in $\mathbb{R}^2$.

Let us first introduce some special partition for bounded sets in $\mathbb{R}^n$.

For a bounded set $\mathbf{E}$, which is Lebesgue measurable in $\mathbb{R}^n$, we write $|\mathbf{E}|$ for its Lebesgue measure.

We will use the letter Q to denote any closed n-dimensional cube in $\mathbb{R}^n$ with its n-faces parallel to the co-ordinate axes. Only such cubes will be used in what follows.

Consider a partition of a cube Q into 2^n equal cubes and repeat this procedure with each new one. By on applying this an infinite number of times, we obtain a system of sub-cubes of Q, denoted for short by $(Q,2)$.

For $P\in(Q,2)$ we define the rank of P by $[P]=n^{-1}\log_2\frac{|Q|}{|P|}$. In this way we will consider the following sets $(k=0,\dots,\infty)$:

$$(\mathbf{E})_k^+=\bigcup\left\{P;\,P\in(Q,2),\ [P]=k,\ |P\cap\mathbf{E}|>\frac{|P|}{2}\right\},$$

$$(\mathbf{E})_k^-=\bigcup\left\{P;\,P\in(Q,2),\ [P]=k,\ |P\cap\mathbf{E}|\leqslant\frac{|P|}{2}\right\},$$

$$(\mathbf{E})_k=\bigcup\big\{P;\,P\in(Q,2),\ [P]=k,\ \big|P\cap(\mathbf{E})_{k+1}^+\big|>0,\ \big|P\cap(\mathbf{E})_{k+1}^-\big|>0\big\}. \tag{3}$$

From now on we assume Γ to be a compact topological surface enclosing a Jordan domain $\Omega\subset\mathbb{R}^n$ and such that $\mathcal{H}^{n-1}(\Gamma)<\infty$.

Definition 1.1. Let $0<\beta\leqslant\alpha<1$. We will say that Γ belongs to the class $\mathcal{V}_{\alpha,\beta}^{(n)}$ (we write $\Gamma\in\mathcal{V}_{\alpha,\beta}^{(n)}$ for short) if there exists a constant c such that for any cube Q

$$\sum_{k=0}^{\infty} 2^{k(1-\alpha)}\left|(\Omega\cap Q)_k\right| \leqslant c|Q|^{1+\frac{\beta-\alpha}{n}}, \tag{4}$$

where $(\Omega\cap Q)_k$ is the system (3) with $\mathbf{E}=\Omega\cap Q$.

In order to get some control on the size of the lower density of Γ, we will need the following assumption: there is a constant $c>0$ such that

$$cr^{n-1} \leqslant \mathcal{H}^{n-1}\big(\Gamma\cap B(x,r)\big) \quad \text{for } x\in\Gamma,\ 0<r\leqslant d:=\operatorname{diam}(\Gamma). \tag{5}$$

This lower bound is typically derivable from topological assumptions; note that it is automatically fulfilled for curves and also for any 1-dimensional compact connected set.

1.2. Basic geometric remarks

It is worth pointing out that there does exist a deep relationship between the aforementioned conditions (4), (5) and the AD-regularity, to know: (i) If Γ is AD-regular, then $\Gamma\in\mathcal{V}^{(n)}_{\alpha,\alpha}$ for any $0<\alpha<1$; (ii) If $\frac{n-1}{n}<\alpha<1$, then $\Gamma\in\mathcal{V}^{(n)}_{\alpha,n\alpha-n+1}$ whenever Γ satisfies (5).

2. Main result

In deriving our main result we have made use of the following direct and inverse estimates. The detailed proofs of these will appear elsewhere.

Lemma 2.1. *Let Γ satisfy* (5)*, $x\in\Gamma$, $f\in C^{0,\alpha}(\Gamma)$ and $0<\frac{\varepsilon_2}{2}\leqslant\varepsilon_1<\varepsilon_2<d$. Then we have*

$$\left|\int_{\Gamma(x,\varepsilon_1,\varepsilon_2)} \frac{x-y}{|y-x|^n}\nu(y)f(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\right| \leqslant c\left(\sup_{y\in\Gamma(x,\varepsilon_1,\varepsilon_2)}\left|f(y)\right| + \frac{1}{\varepsilon_1^{n-1}|Q|^{\frac{1-\alpha}{n}}}\sum_{k=0}^{\infty}\left|(\Omega\cap Q)_k\right|2^{k(1-\alpha)}\right),$$

for every cube Q containing $\Gamma(x,\varepsilon_1,\varepsilon_2):=\{\xi\in\Gamma\colon\ \varepsilon_1\leqslant|\xi-x|\leqslant\varepsilon_2\}$.

Lemma 2.2. *Let Γ satisfy* (5)*. Then, for any cube Q with $|Q|^{\frac{1}{n}}\leqslant\frac{d}{5\sqrt{n}}$, there exist $t_1,t_2\in\Gamma$ with $2\sqrt{n}|Q|^{\frac{1}{n}}\leqslant|t_1-t_2|\leqslant 5\sqrt{n}|Q|^{\frac{1}{n}}$ and a function $\mathbf{f}\in C^{0,\alpha}(\Gamma)$ such that*

$$\left|\mathbf{Hf}(t_1)-\mathbf{Hf}(t_2)\right| \geqslant c|Q|^{\frac{1}{n}(\alpha-1)}\sum_{k=0}^{\infty}2^{k(1-\alpha)}\left|(\Omega\cap Q)_k\right|.$$

Lemma 2.3. *Let $x\in\Gamma$. Then, for $0<\varepsilon<d$ there exists a positive constant c such that*

$$\left|\int_{\Gamma(x,\varepsilon,d)} \frac{x-y}{|y-x|^n}\nu(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\right| \leqslant c.$$

We can now formulate our main result:

Theorem 2.4. *Let Γ satisfy* (5)*. Then* H *is bounded in $C^{0,\alpha}(\Gamma)$ if and only if $\Gamma\in\mathcal{V}^{(n)}_{\alpha,\alpha}$.*

Sketch of the Proof. The idea in proving the sufficiency of condition (4) is to split $\mathrm{H}f(x)-\mathrm{H}f(y)$ as

$$\mathrm{H}f(x)-\mathrm{H}f(y) = \int_{\Gamma(x,0,4\varepsilon)} \frac{x-\xi}{\sigma_n|\xi-x|^n}\nu(\xi)\big(f(\xi)-f(x)\big)\,\mathrm{d}\mathcal{H}^{n-1}(\xi)$$
$$- \int_{\Gamma(y,0,5\varepsilon)} \frac{y-\xi}{\sigma_n|\xi-y|^n}\nu(\xi)\big(f(\xi)-f(y)\big)\mu_{x,\varepsilon}(\xi)\,\mathrm{d}\mathcal{H}^{n-1}(\xi)$$

$$
\begin{aligned}
&- \int\limits_{\Gamma(x,2\varepsilon,4\varepsilon)} \frac{y-\xi}{\sigma_n|\xi-y|^n}\nu(\xi)\big(f(\xi)-f(y)\big)\big(1-\mu_{x,\varepsilon}(\xi)\big)\,\mathrm{d}\mathcal{H}^{n-1}(\xi)\\
&+ \int\limits_{\Gamma(x,4\varepsilon,d)} \left(\frac{x-\xi}{\sigma_n|\xi-x|^n}-\frac{y-\xi}{\sigma_n|\xi-y|^n}\right)\nu(\xi)\big(f(\xi)-f(x)\big)\,\mathrm{d}\mathcal{H}^{n-1}(\xi)\\
&- \int\limits_{\Gamma(x,4\varepsilon,d)} \frac{y-\xi}{\sigma_n|\xi-y|^n}\nu(\xi)\big(f(x)-f(y)\big)\,\mathrm{d}\mathcal{H}^{n-1}(\xi)+\big(f(x)-f(y)\big)=\sum_{k=1}^{6} I_k,
\end{aligned}
$$

where $x, y \in \Gamma$, $|x-y| =: \varepsilon > 0$ and

$$
\mu_{x,\varepsilon} := \begin{cases} \min\left\{1, 2-\dfrac{|\xi-x|}{2\varepsilon}\right\} & \text{if } |\xi-x| \leqslant 4\varepsilon,\\ 0 & \text{if } |\xi-x| > 4\varepsilon. \end{cases}
$$

We now apply Lemma 2.1 to estimate the first four truncated integrals in the above decomposition. To estimate the fifth integral we need only consider Lemma 2.3.

Applying (4) we conclude that $|\mathrm{H}f(x)-\mathrm{H}f(y)| \leqslant \sum_{k=1}^{6}|I_k| \leqslant c\|f\|_\alpha|x-y|^\alpha$.

Conversely, if we suppose that H is bounded in $C^{0,\alpha}(\Gamma)$, then a direct application of Lemma 2.2 gives the desired assertion. □

3. Concluding remarks

Finally a few comments on the main result of the article.

The idea of our procedure is in spirit adapted from that applied in [10], (see also [7]). After a quick comparison, one sees immediately that Theorem 2.4 and Theorem 1 in [10] have many formal similarities. However, we claim that the pass to higher dimensions is far from being trivial.

Our method strongly depends on the usage of condition (5) which is automatic and hence hidden for curves in $\mathbb{C}$. This sort of "left AD-regularity" has been previously introduced in [1] and once again allows the efficient use of some minimal covering by balls and the so-called packing number typically employed in geometric measure theory (see [6,8]).

Moreover, in order to prove our generalization, several necessary modifications have been introduced in order to overcome the non-commutativity of the product in a Clifford algebra.

References

[1] R. Abreu Blaya, J. Bory Reyes, T. Moreno García, Cauchy transform on non-rectifiable surfaces in Clifford analysis, J. Math. Anal. Appl. 339 (2008) 31–44.

[2] R. Abreu Blaya, D. Peña Peña, J. Bory Reyes, Clifford Cauchy type integrals on Ahlfors–David regular surfaces in $\mathbb{R}^{m+1}$, Adv. Appl. Clifford Algebras 13 (2) (2003) 133–156.

[3] F. Brackx, R. Delanghe, F. Sommen, Clifford Analysis, Research Notes in Mathematics, vol. 76, Pitman (Advanced Publishing Program), Boston, 1982.

[4] G. David, Opérateurs intégraux singuliers sur certaines courbes du plan complexe (Singular integral operators over certain curves in the complex plane), Ann. Sci. Ecole Norm. Sup. (4) 17 (1) (1984) 157–189 (in French).

[5] G. David, S. Semmes, Analysis of and on Uniformly Rectifiable Sets, Mathematical Surveys and Monographs, vol. 38, American Mathematical Society, Providence, RI, 1993.

[6] H. Federer, Geometric Measure Theory, Die Grundlehren der mathematischen Wissenschaften, vol. 153, Springer-Verlag New York Inc., New York, 1969.

[7] E.G. Guseĭnov, The Plemelj–Privalov theorem for generalized Hölder classes, Mat. Sb. 183 (2) (1992) 21–37 (in Russian); Translation in Russian Acad. Sci. Sb. Math. 75 (1) (1993) 165–182.

[8] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995.

[9] V.V. Salaev, Direct and inverse estimates for a singular Cauchy integral along a closed curve, Math. Notes 19 (1976) 221–231.

[10] V.V. Salaev, E.G. Guseĭnov, R.K. Seĭfullaev, The Plemelj–Privalov theorem, Dokl. Akad. Nauk SSSR 315 (4) (1990) 790–793 (in Russian); Translation in Soviet Math. Dokl. 42 (3) (1991) 849–852.