Functional Analysis

Homology of free quantum groups

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Abstract

We compute the Hochschild homology of the free orthogonal quantum group $A_o(n)$. We show that it satisfies Poincaré duality and should be considered to be a 3-dimensional object. We then use recent results of R. Vergnioux to derive results about the ℓ^2-homology of $A_o(n)$ and estimates on the free entropy dimension of its set of generators. In particular, we show that the ℓ^2 Betti-numbers of $A_o(n)$ all vanish and that the free entropy dimension is less than 1. ***To cite this article: B. Collins et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

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Résumé

Homologie des groupes quantiques libres. Nous calculons l'homologie de Hochschild du groupe quantique libre $A_o(n)$. Nous montrons qu'il vérifie une dualité de Poincaré et qu'il peut être considéré comme un objet en 3 dimensions. Nous utilisons ensuite des résultats récents de R. Vergnioux pour en déduire des résultats sur l'homologie ℓ^2 de $A_o(n)$ et des estimées sur la dimension libre de l'ensemble des générateurs. En particulier, nous démontrons que tous les nombres de Betti ℓ^2 de $A_o(n)$ sont nuls et que la dimension libre de $A_o(n)$ est majorée par 1. ***Pour citer cet article : B. Collins et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

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Soit $A_o(n)$ la $*$-algèbre unitale engendrée par les n^2 éléments auto-adjoints u_{ij}, for $1 \leqslant i, j \leqslant n$ avec les relations :

$$\sum_{j=1}^{n} u_{ij} u_{kj} = \delta_{ik}, \quad \text{et} \quad \sum_{j=1}^{n} u_{ji} u_{jk} = \delta_{ik}.$$

L'algèbre $A_o(n)$ est munie naturellement d'une structure d'algèbre de Kac et rentre dans le cadre des groupes quantiques compacts introduits par Woronowicz dans [11]. Ce groupe a été introduit par Wang dans [10] et sa théorie de la représentation a été étudiée par Banica dans [1]. Vaes et Vergnioux ont démontré dans [7] que l'algèbre de von Neumann $A_o(n)''$ obtenue par complétion de $A_o(n)$ par rapport à son état tracial de Haar μ est en fait un facteur solide et plein. Vergnioux a démontré récemment [9] que le premier nombre de Betti ℓ^2 de $A_o(n)$ est nul, ce qui laisse penser que $A_o(n)''$ n'est pas isomorphe à un facteur du groupe libre.

Nous démontrons que en fait tous les nombre de Betti ℓ^2 de $A_o(n)$ sont nuls. Pour ce faire, nous calculons une résolution libre de $A_o(n)$, observons qu'elle est auto-duale, et en déduisons à l'aide d'un résultat de Thom [6], que $\beta_1^{(2)}(A_o(n)) = \beta_2^{(2)}(A_o(n))$ et que $\beta_k^{(2)}(A_o(n)) = 0$ pour tout $k \geqslant 3$.

La vérification de l'exactitude de la résolution est un exercice d'algèbre relativement simple au niveau de toutes les flèches, sauf au niveau de $\ker \phi_2 = \operatorname{im} \phi_1$ (voir le Théorème 1.1 pour le résultat et les notations), où nous n'avons pas trouvé de preuve élémentaire. Nous avons eu recours à la théorie des bases de Gröbner et les détails se trouvent dans [3], ainsi que dans la thèse de l'un des auteurs (J. Härtel).

1. Introduction and main results

Let $A_o(n)$ be the unital $*$-algebra generated by self-adjoint elements u_{ij}, $1 \leqslant i, j \leqslant n$, with relations:

$$\sum_{j=1}^n u_{ij}u_{kj} = \delta_{ik}, \quad \text{and} \quad \sum_{j=1}^n u_{ji}u_{jk} = \delta_{ik}.$$

We call this algebra the *free orthogonal quantum group*, associated to the integer $n \in \mathbb{N}$. Its abelianization is naturally isomorphic to the algebra of polynomial functions on the compact Lie group $O(n)$. It is endowed with the coproduct $\Delta u_{ij} = \sum_k u_{ik} \otimes u_{kj}$, the counit $\varepsilon(u_{ij}) = \delta_{ij}$ and the antipode $S(u_{ij}) = u_{ji}$. The maps $\Delta : A_o(n) \to A_o(n) \otimes A_o(n)$, $\varepsilon : A_o(n) \to \mathbb{C}$ and $S : A_o(n) \to A_o(n)^{op}$ are algebra morphisms, therefore they are fully determined by the above relations.

We are in the setting of compact quantum matrix group of Woronowicz [11] therefore there exists a unique left and right invariant Haar state, which we denote μ. Since $S^2 = \operatorname{id}$, it follows from Woronowicz's theory that μ is a finite trace on $A_o(n)$. It has been proved by Woronowicz that this state is faithful on $A_o(n)$, and it was proved in [7] that the von Neumann algebra obtained by the GNS completion of $A_o(n)$ in μ, denoted by $A_o(n)''$, is actually a II_1 factor for $n \geqslant 3$ and that this factor is full and solid. These properties are shared by the von Neumann algebra of the free group, therefore it is an interesting question whether $A_o(n)''$ is isomorphic to an interpolated free group factor. Although we are not able to answer this question, we observe that unlike in the free group case, all ℓ^2 Betti numbers vanish. This relies on recent of work of R. Vergnioux [9], who showed that the first ℓ^2 Betti number of $A_o(n)$ vanishes. As far as we know, this is the first example of complete computation of ℓ^2 Betti numbers for a quantum group beyond the case of amenable quantum groups (cf. Kyed, [5]).

We also observe that a corollary of Vergnioux's results following Connes and Shlyakhtenko [2] is that the non-microstates free entropy dimension of $(A_o(n), \mu)$ is less than 1, and this follows from the fact that its zeroeth and first ℓ^2 Betti numbers vanish. The problem of computing the microstates free entropy dimension of $(A_o(n), \mu)$ is still open, as we do not know whether the family u_{ij} has microstates.

By a slight abuse notation, δ_{ij} will denote either an element of $\mathbb{C}$ or a multiple of the identity in $A_o(n)$ depending on the situation. We keep this notation as there will be no ambiguity. Let M_n be the $n \times n$ complex matrix algebra over $\mathbb{C}$. If N is a $\mathbb{C}$-vector space we denote by $M_n N$ the algebraic tensor product of M_n and N over $\mathbb{C}$. Our first main result is the following:

Theorem 1.1. *Let $n \geqslant 1$ and $A_o(n)$ be the free orthogonal quantum group. Let e_{ij} be the canonical matrix basis of the $n \times n$ matrices and $r_{ik} = 1 \otimes e_{ik} + \sum_j u_{ij} \otimes e_{kj} \in M_n A_o(n)$. Then the following sequence C_* of $A_o(n)$ modules*

$$0 \to A_o(n) \xrightarrow{\phi_1} M_n A_o(n) \xrightarrow{\phi_2} M_n A_o(n) \xrightarrow{\phi_3} A_o(n) \xrightarrow{\varepsilon} \mathbb{C} \to 0$$

with maps ϕ_1, ϕ_2, ϕ_3 defined on the respective module generators with

$$\phi_1(1) = \sum_{i,j}(u_{ij} - \delta_{ij}) \otimes e_{ij}, \quad \phi_2(e_{ij}) = r_{ij} \quad and \quad \phi_3(e_{ij}) = u_{ij} - \delta_{ij}$$

yield a free resolution of the co-unit.

We describe some details of the proof in Section 2. Our second main result is the following:

Theorem 1.2. *All ℓ^2 Betti numbers of $A_o(n)$ vanish. In particular, the free entropy dimension of the generators of $A_o(n)$ is* 1.

We need three ingredients to complete the proof of Theorem 1.2. First we need Theorem 1.1; then we need to observe that our resolution has a self-duality property (Theorems 3.1 and 3.2), also known as Poincaré duality, which allows us to use a result of Thom [6] to identify the Betti numbers pairwise. This is done is Section 3. And last we need the recent result of Vergnioux [9] mentioned above (Section 4).

2. Description of the resolution of $A_o(n)$

The surjectivity of ε and the injectivity of ϕ_1 are obvious. First of all, let us show that $\mathrm{im}(\phi_3) = \ker(\varepsilon)$. The inclusion $\mathrm{im}(\phi_3) \subset \ker(\varepsilon)$ is obvious, so we focus on the proof of $\ker(\varepsilon) \subset \mathrm{im}(\phi_3)$. Let $a = \sum_j a_j \in A_o(n)$ be a sum of monomials with $\varepsilon(a) = 0$, where a monomial is understood as a product of u_{ij}'s. This implies that $\sum_j \varepsilon(a_j) = 0$ and hence $a = \sum_j (a_j - \varepsilon(a_j))$. Since $\varepsilon(a_j - \varepsilon(a_j)) = 0$, it is enough to prove that if a is a monomial then $a - \varepsilon(a) \cdot 1 \in \mathrm{im}(\phi_3)$. If a is of length 0, there is nothing to prove. If $a = a' u_{ij}$ for some $1 \leqslant i, j \leqslant n$, then $a = a'(u_{ij} - \delta_{ij}) + a'\delta_{ij}$. However, a' has smaller length than a. This proves, that we can inductively reduce the length of any monomial to 0. Hence $\ker(\varepsilon) = \mathrm{im}(\phi_3)$.

We continue to show that $\ker(\phi_3) = \mathrm{im}(\phi_2)$. For this purpose, we let $F = \mathbb{C}\langle u_{ij} \rangle$ and set

$$R_{ik} = \sum_{j=1}^{n} u_{ij} u_{kj} - \delta_{ik} \in F \quad \text{and} \quad L_{ik} = \sum_{j=1}^{n} u_{ji} u_{jk} - \delta_{ik} \in F.$$

Moreover, we set

$$r_{ik} = 1 \otimes e_{ik} + \sum_j u_{ij} \otimes e_{kj} \in M_n A_o(n) \quad \text{and} \quad l_{ik} = 1 \otimes e_{ki} + \sum_j u_{ji} \otimes e_{jk} \in M_n A_o(n).$$

We start with the first intermediate lemma:

Lemma 2.1. *For all j, k, the following holds true*:

$$r_{jk} = \sum_i u_{ji} l_{ik} \quad \text{and} \quad l_{jk} = \sum_i u_{ij} r_{ik}.$$

Indeed, we compute

$$r_{jk} = \sum_i u_{ji} \otimes e_{ki} + \sum_m \delta_{jm} \otimes e_{mk} = \sum_i u_{ji} \otimes e_{ki} + \sum_{i,m} u_{ji} u_{mi} \otimes e_{mk} = \sum_i u_{ji} l_{ik} \in M_n A_o(n).$$

The other computation is similar. This finishes the proof of Lemma 2.1. We also need the following lemma:

Lemma 2.2. *The linear map $\tilde{\phi}_3 : M_n(F) \to F$, which is defined by*

$$(a_{ij})_{i,j=1}^{n} \xrightarrow{\tilde{\phi}_3} \sum_{ij} a_{ij}(u_{ij} - \delta_{ij}),$$

is injective.

The proof of this lemma is elementary: assume that $\sum_{ij} a_{ij}(u_{ij} - \delta_{ij}) = 0$. By considering the degree in u_{ij} this implies that $a_{ij} = 0$ and this holds true for all i, j.

Now we prove that $\ker(\phi_3) = \mathrm{im}(\phi_2)$. Let $a = (a_{ij})_{i,j=1}^{n}$ by any element of $M_n A_o(n)$ which maps to zero under ϕ_3. Take any lift $\tilde{a} \in M_n F$ of $a \in M_n A_o(n)$. Since a maps to $0 \in A_o(n)$, $\tilde{\phi}_3(\tilde{a})$ has to have a representation

$$\tilde{\phi}_3(\tilde{a}) = \sum_\alpha f_\alpha R_{i_\alpha j_\alpha} f'_\alpha + \sum_\beta g_\beta L_{i_\beta j_\beta} g'_\beta,$$

for some $f_\alpha, f'_\alpha, g_\beta, g'_\beta \in F$ and $1 \leqslant i_\alpha, j_\alpha, i_\beta, j_\beta \leqslant n$. We will construct a pre-image of $\tilde{\phi}_3(\tilde{a})$ by constructing a pre-image of each of the summands. If $f'_\alpha = 1$, then $f_\alpha r_{i_\alpha j_\alpha}$ is a pre-image of $f_\alpha R_{i_\alpha j_\alpha}$. However, if $f'_\alpha = f''_\alpha u_{ij}$, then

$$f_\alpha R_{i_\alpha j_\alpha} f'_\alpha = f_\alpha R_{i_\alpha j_\alpha} f''_\alpha (u_{ij} - \delta_{ij}) + \delta_{ij} f_\alpha R_{i_\alpha j_\alpha} f''_\alpha.$$

Hence we reduced the length of f'_α by one on the expense of an additional summand. However, this additional summand has a pre-image and hence the claim follows by induction. Since the map $\tilde{\phi}_3$ is injective by Lemma 2.2, we actually reconstructed $\tilde{a}$ which is now by construction of the form:

$$\tilde{a} = \sum_\gamma h_\gamma R_{i_\gamma j_\gamma} h'_\gamma + \sum_\delta k_\delta R_{i_\delta j_\delta} k'_\delta + \sum_{i,k} p_{ik} r_{ik} + s_{ik} l_{ik},$$

for some $h_\gamma, k_\delta, p_{ik}, s_{ik} \in F$, $h'_\gamma, k'_\delta \in M_n(F)$ and $1 \leqslant i_\gamma, j_\gamma, i_\delta, j_\delta \leqslant n$. Taking the class of $\tilde{a}$ in $M_n A_o(n)$ which is just a, we see, using Lemma 2.1, that $a \in \mathrm{im}(\phi_2)$. Since $a \in \ker(\phi_3)$ was arbitrary it follows that $\ker(\phi_3) \subset \mathrm{im}(\phi_2)$. Again, the reverse inclusion is obvious. This implies that $\ker(\phi_3) = \mathrm{im}(\phi_2)$.

It remains to show that $\ker \phi_2 = \mathrm{im} \phi_1$. The proof of this claim is more involved and uses a theory of Gröbner basis for non-commutative rings. The details can be found in the PhD dissertation of the second named author [3].

3. Poincaré duality and computation of the ℓ^2 Betti numbers

In this section we first recall the basic notions of Poincaré duality for non-commutative rings, which was developed in [8]. For a nice survey and some background we refer to [4]. In our setup, we can show directly that the resolution is self-dual; indeed $\phi_1^t = \phi_3$ and $\phi_2^t(e_{ij}) = l_{ji}$ as one can easily show. This is sufficient to construct an explicit isomorphism $C_{3-*} \cong \hom_{A_o(n)}(C_*, A_o(n))$, where $C_* \to \mathbb{C}$ denotes the resolution from Theorem 1.1.

Definition 3.1. Let $(A, \Delta, S, \varepsilon, \eta)$ be a Hopf-algebra over $\mathbb{C}$. We define

$$H_*(A, M) = \mathrm{Tor}_*^A(\mathbb{C}, M), \quad \text{and} \quad H^*(A, M) = \mathrm{Ext}_A^*(\mathbb{C}, M),$$

where we consider $\mathbb{C}$ as the trivial right A-module via the co-unit. Here, $\mathrm{Tor}_*^A(A, ?)$ denotes the left-derived functor of the relative tensor product over A. Similarly, $\mathrm{Ext}_A^*(A, ?)$ denotes the right-derived functor of $\hom_A(A, ?)$.

As usual, we set $A^e = A \otimes_\mathbb{C} A^{\mathrm{op}}$. Since A is a Hopf-algebra $(1 \otimes S) \circ \Delta : A \to A^e$ provides a natural embedding. Note that $? \otimes_A A^e$ is an exact functor from A-modules to A-bi-modules which sends the trivial representation to the trivial bi-module and free modules to free bi-modules. Hence, our resolution of the co-unit provides at the same time a free resolution of the bi-module A by free A^e-modules. Hence, it is equivalent to consider generalized group (co)homology in the sense above and Hochschild (co)homology of the Hopf algebra. We state our results only in terms of the objects introduced in the previous definition.

Theorem 3.1. *Let $n \geqslant 2$ and $A_o(n)$ be the free orthogonal quantum group.*

(i) $\qquad \mathrm{Ext}_{A_o(n)}^*(\mathbb{C}, A_o(n)) = \begin{cases} \mathbb{C}, & * = 3, \\ 0, & * \neq 3. \end{cases}$

(ii) *There is a natural isomorphism*

$$H_*(A_o(n), M) \cong H^{3-*}(A_o(n), M)$$

for any $A_o(n)$-module M.

(iii) *$A_o(n)$ is smooth with $\dim A = 3$.*

Assertion (i) follows straight from the observation the resolution $C_* \to \mathbb{C}$ is self-dual. Indeed, the cohomology of the same complex (but upside-down) is clearly $\mathbb{C}$ in dimension 3. The proof of assertion (ii) uses the fact that the $A_o(n)$-modules in the resolution $C_* \to \mathbb{C}$ are finitely generated free. Hence,

$$\hom_{A_o(n)}(C_*, M) \cong \hom_{A_o(n)}(C_*, A_o(n)) \otimes_{A_o(n)} M.$$

The left side computes cohomology (by definition) and the right side computes homology since C_* is self-dual, i.e. $\hom_{A_o(n)}(C_*, A_o(n)) \cong C_{3-*}$.

Theorem 3.2. *Let $A_o(n)$ be the free orthogonal quantum group.*

$$H_*\big(A_o(n), \mathbb{C}\big) = \begin{cases} \mathbb{C}, & *=0, \\ \mathbb{C}^{\oplus \frac{n(n-1)}{2}}, & *=1, \\ \mathbb{C}^{\oplus \frac{n(n-1)}{2}}, & *=2, \\ \mathbb{C}, & *=3, \\ 0, & *\geqslant 4. \end{cases} \tag{1}$$

Recall that $\mathrm{Tor}_*^{A_o(n)}(\mathbb{C}, \mathbb{C})$ can be computed by replacing $\mathbb{C}$ with a projective resolution of it, tensoring over $A_o(n)$ with $\mathbb{C}$, and computing the homology. The result can be easily read off the resulting complex, which is the following:

$$0 \to \mathbb{C} \xrightarrow{\phi_1} M_n\mathbb{C} \xrightarrow{\phi_2} M_n\mathbb{C} \xrightarrow{\phi_3} \mathbb{C} \to 0,$$

with

$$\phi_2\big((a_{ij})_{i,j=1}^n\big) = \sum_{ij} a_{ij}(e_{ik} + e_{ki}) \quad \text{and} \quad \phi_1 = \phi_3 = 0.$$

Clearly, $\phi_2(A) = A + A^t$ in matrix notation, so that the kernel consists of skew-symmetric and the image of all symmetric matrices. This finishes the proof of Theorems 3.1 and 3.2.

4. ℓ^2 homology and free entropy dimension

We recall that the ℓ^2 Betti numbers of $A_o(n)$ are defined as the dimension of the ℓ^2-homology of the quantum group $H_*(A_o(n); \ell^2(A_0(n), \mu))$. It was shown in [6] that the dimension of the cohomology in the same degree agrees with the ℓ^2 Betti number. Theorems 3.1 and 3.2 imply that $\beta_1^{(2)}(A_o(n), \mu) = \beta_2^{(2)}(A_o(n), \mu)$, and $\beta_k^{(2)}(A_o(n), \mu) = 0$ for all $k \geqslant 3$. These assertions follow from Poincaré duality and $\beta_0^{(2)}(A_o(n), \mu) = 0$. The latter claim follows since $L(A_0(n), \tau)$ is diffuse for $n \geqslant 2$.

It has been shown recently by R. Vergnioux in [9] that $\beta_1^{(2)}(A_o(n)) = 0$. His methods uses quantum Cayley graphs. From his result we can deduce that $\beta_k^{(2)}(A_o(n)) = 0$ for all $k \geqslant 3$. It has been proved by Connes and Shlyakhtenko in [2] that the microstate free entropy dimension of any unital $*$-algebra A with a trace τ satisfies $\delta^*(A, \tau) \leqslant \beta_1^{(2)}(A, \tau) - \beta_0^{(2)}(A, \tau) + 1$. This implies in our case that $\delta^*(A_o(n), \mu) = 1$. We do not know whether $A_o(n)$ has microstates and believe that it is an interesting question. If it does, then $\delta(A_o(n), \mu) = \delta^*(A_o(n), \mu) = 1$.

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