



Algebraic Geometry

Some examples of vector bundles in the base locus of the generalized theta divisor

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Abstract

We show that, on the moduli space of semi-stable vector bundles of fixed rank and determinant (of any degree) on a curve, the base locus of the theta divisor as well as its n -multiples is large. This extends known results for the case of trivial determinant and $n = 1$. **To cite this article:** *S. Casalaina-Martin et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).*

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Résumé

Quelques exemples de fibrés dans le lieu de base du diviseur thêta généralisé. On prouve que, sur l'espace de modules des fibrés stables de déterminant fixé (et degré quelconque) sur une courbe, le lieu de base du diviseur thêta généralisé ainsi que ses n -multiples est assez grand. Ce travail étend des résultats connus pour le cas de degré zéro et $n = 1$. **Pour citer cet article :** *S. Casalaina-Martin et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).*

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1. Introduction

Let C be a smooth projective curve over \mathbb{C} of genus $g(C) = g \geq 2$. Fix $r, d \in \mathbb{Z}$ and $r \geq 1$. Consider the moduli space $\mathcal{U}(r, d)$ of (equivalence classes of) semi-stable vector bundles on C of rank r and degree d . For a line bundle $L \in \text{Pic}^d(C)$, consider the subspace $S\mathcal{U}(r, L) \subset \mathcal{U}(r, d)$ of vector bundles with determinant L . Note that because of the isomorphisms given by tensoring with a line bundle, there are at most r not necessarily isomorphic moduli spaces $\mathcal{U}(r, d)$. Therefore, it suffices to consider the slope modulo integers. In the case of fixed determinant, only the degree of L is relevant.

There is an ample line bundle \mathcal{L} on $S\mathcal{U}(r, L)$, called the determinant line bundle, such that $\text{Pic}(S\mathcal{U}(r, L)) = \mathbb{Z}\langle \mathcal{L} \rangle$ [3]. We will be interested in the base locus of the linear system $|\mathcal{L}^{\otimes n}|$ on $S\mathcal{U}(r, L)$, for positive integers n . When $d = 0$, (or equivalently when r divides d), a great deal is known about this base locus. In [10], Raynaud provided

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a finite number of vector bundles in the base locus for very high rank with respect to the genus. Later, Popa [8] showed that there are positive dimensional families of such examples. Some improvements on those results were obtained in [1,4]. Finally, [5] gave, at least theoretically, a characterization of all the points in the base locus (see also [6]).

In sharp contrast with the degree zero case, not much has been done for other degrees, the main reason being that a characterization of fixed points was not available till the Strange Duality Conjecture was proved in [7] (see also [2]). We generalize these results in two directions. We prove that, for any slope, one can find a positive dimensional family in the base locus of the theta divisor in suitable rank. We also show that the base loci of $|\mathcal{L}^{\otimes n}|$ with $n > 1$ are often quite large. Some remarks in this direction were provided in [8].

Theorem 1.1. *Fix a slope $\mu \in \mathbb{Q}$, and a level $n \in \mathbb{N}$. For $N \in \mathbb{N}$ and $L \in \text{Pic}^{\mu N}(C)$, the base locus of the linear system $|\mathcal{L}^{\otimes n}|$ on $S\mathcal{U}(N, L)$ is large provided $N \gg 0$ and $g \gg 0$.*

For more precise conditions on N and g , and for estimates on the dimension of the base loci, see Theorem 3.5 and Theorem 3.7. We point out that Theorem 8.1 in [9] states that on $S\mathcal{U}(N, \mathcal{O}_C)$, $\mathcal{L}^{\otimes n}$ is globally generated for $n \geq \lfloor \frac{N^2}{4} \rfloor$, and it has been conjectured to be for $n \geq N - 1$. The bound obtained in these notes is far from this.

The main idea is to construct explicit examples of vector bundles as in [4,8] by taking wedge powers of certain vector bundles obtained as kernels of an evaluation map. A technical point (cf. Proposition 3.1) using results in [11] and [7] shows that these are in the base locus.

2. Preliminaries

Given $L \in \text{Pic}^d(C)$ and $F \in \mathcal{U}(r_F, d_F)$ such that $\mu(\xi \otimes F) = g - 1$ for $\xi \in S\mathcal{U}(r, L)$ set $\Theta_F = \{\xi \in S\mathcal{U}(r, L) : h^0(\xi \otimes F) > 0\}$. This definition of Θ_F depends on the choice of suitable r, L that, for simplicity, are not included in the notation.

From $\mu(\xi \otimes F) = g - 1$, $r_F = n \frac{r}{\gcd(r, d)}$ for some $n \in \mathbb{N}$. It is well known that if Θ_F is a divisor, then $\Theta_F \in |\mathcal{L}^{\otimes n}|$. The Strange Duality Theorem (Theorem 4 [7]) implies that the linear system $|\mathcal{L}^{\otimes n}|$ is spanned by such divisors, and thus its base locus equals

$$\{\xi \in S\mathcal{U}(r, L) : h^0(F \otimes \xi) > 0 \forall F \in \mathcal{U}(r_F, d_F)\}.$$

Lemma 2.1. *Let $L \in \text{Pic}^d(C)$, and suppose that on $S\mathcal{U}(r, L)$ the base locus of $|\mathcal{L}^{\otimes n}|$ has dimension b . If $r' \geq r$, $L' \in \text{Pic}^{d'}(C)$, and $\frac{d}{r} - \frac{d'}{r'} \in \mathbb{Z}$, then the base locus of $|\mathcal{L}'^{\otimes n}|$ on $S\mathcal{U}(r', L')$ has dimension at least $b + (r' - r)^2(g - 1) + 1 - \delta_{r, r'}$, where $\delta_{r, r'}$ is the Kronecker delta.*

Proof. It suffices to consider slopes up to integers. If $\frac{d}{r} = \frac{d'}{r'}$, consider direct sums of vector bundles of slope $\frac{d}{r}$, and of the appropriate ranks (see [4] Corollary 5.3). \square

Given a globally generated vector bundle E on C , define a vector bundle M_E as the kernel of the evaluation map

$$0 \rightarrow M_E \rightarrow H^0(E) \otimes \mathcal{O}_C \rightarrow E \rightarrow 0.$$

From [4] Theorems 1.1 and 1.2, if $d_E - 2r_E g > -g$ and $0 < i < d_E - r_E g$, the map given by $E \mapsto \wedge^i M_E^\vee$ is generically finite. Below, we give some conditions that insure that for each $E \in \mathcal{U}(r_E, L)$, the corresponding $\wedge^i M_E^\vee$ is in the base locus of $|\mathcal{L}^{\otimes n}|$ on $S\mathcal{U}(\binom{d_E - r_E g}{i}, L^{\otimes \binom{d_E - r_E g - 1}{i - 1}})$ for certain n .

3. Examples

Proposition 3.1. *If $E \in \mathcal{U}(r_E, d_E)$, and $r_F, i \in \mathbb{N}$ satisfy:*

- (1) $\frac{r_F i d_E}{d_E - r_E g} \in \mathbb{N}$,
- (2) $\frac{g-1}{g} \cdot \frac{d_E - r_E g}{r_E r_F} \geq i \geq g$,

then setting $d_F = r_F(g - 1) - \frac{r_F id_E}{d_E - r_E g}$,

$$\Theta_{\wedge^i M_E^\vee} := \{F \in \mathcal{U}(r_F, d_F) : h^0(F \otimes \wedge^i M_E^\vee) > 0\} = \mathcal{U}(r_F, d_F).$$

Proof. Condition (2) above implies that $d_E - r_E g > i > 0$ so $\wedge^i M_E^\vee$ is well defined. From (1) the definition of d_F makes sense and for $F \in \mathcal{U}(r_F, d_F)$, $\mu(\wedge^i M_E^\vee \otimes F) = g - 1$.

From [4], Proposition 5.1, for a generic effective divisor D_i of degree i , there is an immersion $\mathcal{O}(D_i) \rightarrow \wedge^i M_E^\vee$. Hence, it is enough to show that a general vector bundle $F \in \mathcal{U}(r_F, d_F)$ can be written as an extension of vector bundles $0 \rightarrow \mathcal{O}_C(-D_i) \rightarrow F \rightarrow \bar{F} \rightarrow 0$. Since $i \geq g$, $\mathcal{O}_C(-D_i)$ is a general line bundle of degree $-i$. From [11] a general $F \in \mathcal{U}(r_F, d_F)$ can be written as such an extension so long as $\mu(\bar{F}) - \mu(\mathcal{O}_C(-D_i)) \geq g - 1$. From the short exact sequence above and the definition of d_F ,

$$\mu(\bar{F}) - \mu(\mathcal{O}_C(-D_i)) = \frac{r_F(g - 1 - id_E/(d_E - r_E g)) + i}{r_F - 1}.$$

Condition (2) ensures that the needed inequality is satisfied. \square

Corollary 3.2. If $E \in \mathcal{U}(r_E, d_E)$ with $\mu(E) = (\frac{a}{b})g$ for some $a, b \in \mathbb{N}$, and $r_F, i \in \mathbb{N}$ satisfy the following conditions:

$$(1) r_F i \frac{a}{a-b} \in \mathbb{N}, \quad (2) (g - 1) \frac{a}{b} \geq r_F i \frac{a}{a-b} \geq r_F g \frac{a}{a-b},$$

then, setting $d_F = r_F(g - 1) - r_F i \frac{a}{a-b}$,

$$\Theta_{\wedge^i M_E^\vee} := \{F \in \mathcal{U}(r_F, d_F) : h^0(F \otimes \wedge^i M_E^\vee) > 0\} = \mathcal{U}(r_F, d_F).$$

Remark 3.3. With the assumptions in the corollary, $\mu(\wedge^i M_E^\vee) = \frac{ia}{a-b} \equiv \frac{ib}{a-b} \pmod{\mathbb{Z}}$. Set $r = \text{rank}(\wedge^i M_E^\vee)$, and $d = \text{deg}(\wedge^i M_E^\vee)$. It follows that $\frac{r}{\text{gcd}(r,d)} = \frac{a-b}{\text{gcd}(a-b,ia)}$. Set $n = r_F \cdot \frac{\text{gcd}(a-b,ia)}{a-b}$. For any $L \in \text{Pic}^d(C)$ and an appropriate line bundle η on C , it follows that $\wedge^i M_{E \otimes \eta}^\vee \in S\mathcal{U}(r, L)$ is a base point for the linear system $|\mathcal{L}^{\otimes n}|$ on $S\mathcal{U}(r, L)$. In other words, with the assumptions above $\wedge^i M_{E \otimes \eta}^\vee \in \Theta_F \subseteq S\mathcal{U}(r, L) \forall F \in \mathcal{U}(r_F, d_F)$.

Proposition 3.4. Fix $r_E, r_F, A, b, i' \in \mathbb{N}$ satisfying the following conditions:

- (1) $b | r_E g$;
- (2) $\frac{g-1}{b} \geq i' \geq \frac{g}{A}$;
- (3) $r_F \geq \frac{b}{A}$.

Set $n = \text{gcd}(r_F, i'b)$, $r = (\frac{Ag}{b} r_F r_E)$, $d = r(i'A + \frac{i'b}{r_F})$, and fix a line bundle $L \in \text{Pic}^d(C)$. On $S\mathcal{U}(r, L)$ the dimension of the base locus of $|\mathcal{L}^{\otimes n}|$ is at least

$$(r_E^2 - 1)(g - 1).$$

Proof. Take $a = Ar_F + b$, $i = i'A$ in the previous corollary. Assumption (3) in the proposition insures that the map taking $E \mapsto M_E$ is generically finite [4]. \square

Theorem 3.5. Let $\mu \in \mathbb{Q}$, and suppose there exist numbers $i', r_F \in \mathbb{N}$ with $1 \leq i' \leq g - 1$ such that $\mu \equiv \frac{i'}{r_F} \pmod{\mathbb{Z}}$. Suppose that $N, r_E \in \mathbb{N}$, $r_E \geq 2$, and $N \geq N_0 := (\frac{r_F r_E g^2}{i' g})$. For $L \in \text{Pic}^{\mu N}(C)$, on $S\mathcal{U}(N, L)$, the base locus of the linear system $|\mathcal{L}|$ has dimension at least $((r_E^2 - 1) + (N - N_0)^2)(g - 1) + 1 - \delta_{N, N_0}$. The same can be said for the linear system $|\mathcal{L}^{\otimes \text{gcd}(r_F, i')}|$.

Proof. Take $A = g$ and $b = 1$ in Proposition 3.4. \square

Remark 3.6. For a given slope $\mu \in \mathbb{Q}$, taking a curve of high enough genus, the theorem implies that the base locus is nonempty provided the rank is sufficiently large. In particular, Theorem 1.1 follows from Theorem 3.5.

Taking $A = 2$, $b = 1$, and $i' = r_F = g - 1$ in Proposition 3.4, we get examples with lower rank and higher level.

Theorem 3.7. Suppose that $N, r_E \in \mathbb{N}$, $r_E \geq 2$, and

$$N \geq N_0 := \binom{2g(g-1)r_E}{2(g-1)}.$$

Then on $S\mathcal{U}(N, \mathcal{O}_C)$, the base locus of the linear system $|\mathcal{L}^{\otimes g-1}|$ has dimension at least

$$\left((r_E^2 - 1) + (N - N_0)^2\right)(g - 1) + 1 - \delta_{N, N_0}.$$

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