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A congruence theorem for minimal surfaces in S^5 with constant contact angle

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Abstract

We provide a congruence theorem for minimal surfaces in S^5 with constant contact angle using the Gauss–Codazzi–Ricci equations. More precisely, we prove that the Gauss–Codazzi–Ricci equations for minimal surfaces in S^5 with constant contact angle satisfy an equation for the Laplacian of the holomorphic angle. ***To cite this article: R.R. Montes, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Théorème de congruence pour les surfaces minimales en S^5 avec angle de contact constant. Nous présentons un théorème de congruence pour les surfaces minimales en S^5 avec angle de contact constant en utilisant les équations de Gauss–Codazzi–Ricci. Plus précisémént, nous prouvons que les équations de Gauss–Codazzi–Ricci pour les surfaces minimales en S^5 avec angle de contact constant satisfont une équation pour le Laplacien de l'angle holomorphe. ***Pour citer cet article : R.R. Montes, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

We will establish a condition in order to investigate minimal surfaces in S^5 with constant contact angle using the Gauss–Codazzi–Ricci equations. We define α to be the angle given by $\cos\alpha = \langle ie_1, v\rangle$, where e_1 and v are defined in Section 2. The holomorphic angle α is the analogue of the Kähler angle introduced by Chern and Wolfson in [2].

Recently, in [4], we construct a family of minimal tori in S^5 with constant contact angle and constant holomorphic angle. These tori are parametrized by the following circle equation:

$$a^2 + \left(b - \frac{\cos\beta}{1+\sin^2\beta}\right)^2 = 2\frac{\sin^4\beta}{(1+\sin^2\beta)^2}. \tag{1}$$

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In particular, when $a = 0$ in (1), we recover the examples found by Kenmotsu, in [3]. These examples are defined for $0 < \beta < \frac{\pi}{2}$. Also, when $b = 0$ in (1), we find a new family of minimal tori in S^5, and these tori are defined for $\frac{\pi}{4} < \beta < \frac{\pi}{2}$. Moreover, we show that the Gaussian curvature K of a minimal surface in S^5 with contact angle β and holomorphic angle α is given by:

$$K = -\left(1 + \tan^2\beta\right)|\nabla\beta|^2 - \tan\beta\Delta\beta - 2\cos\alpha\left(1 + 2\tan^2\beta\right)\beta_1 + 2\tan\beta\sin\alpha\alpha_1 - 4\tan^2\beta\cos^2\alpha. \tag{2}$$

In this Note, we will establish a congruence theorem for minimal surfaces in S^5 with constant contact angle (β) with $(0 < \beta < \frac{\pi}{2})$. Supposing that the second fundamental form (Π_3) in the direction e_3 is diagonalized, and using Gauss–Codazzi–Ricci equations, we prove the following theorem:

Theorem 1. *Consider S a simply connected riemannian surface, $\alpha : S \to]0, \frac{\pi}{2}[$ a function over S that verifies the following equation*:

$$\begin{aligned} \Delta(\alpha) = {} & \cot\alpha\csc^3(\beta)|\nabla\alpha|^2 + a^2\cot\alpha\cot^4(\beta) - 2a\cot\alpha\csc\beta\cot^2\beta\alpha_2 \\ & - 2\cos\alpha(\cot\beta - \tan\beta)\tan^2\beta\alpha_1 + \sin\alpha\cos\alpha\left(5 - \cot^4\beta - 3\csc^2\beta\right) \end{aligned} \tag{3}$$

then there exist one and only one minimal immersion of S into S^5 with constant contact angle (β) such that (α) is the holomorphic angle of this immersion, where **a** *is given in Section* 2, *and its determined as a function of α and β in Section* 3.

As an immediate consequence of this method, we have a classification of certain flat minimal surfaces in S^5

Corollary 1. *Suppose that the contact angle (β) is constant, suppose that S is a flat minimal surface in S^5 with constant principal curvature in the direction e_3, that is,* **a** *is constant, and (Π_3) is diagonalized, then the holomorphic angle (α) must be constant.*

2. Gauss–Codazzi–Ricci equations for a minimal surface in S^5 with constant contact angle β

In this section, we will compute the Gauss–Codazzi–Ricci equations for a minimal surface in S^5 with constant contact angle β. Let S be an immersed orientable surface in S^5. Consider in $\mathbb{C}^3$ the following objects:

- the Hermitian product: $(z, w) = \sum_{j=0}^{2} z^j \bar{w}^j$;
- the inner product: $\langle z, w\rangle = \mathrm{Re}(z, w)$;
- the unit sphere: $S^5 = \{z \in \mathbb{C}^3 \mid (z, z) = 1\}$;
- the *Reeb* vector field in S^5, given by: $\xi(z) = iz$;
- the contact distribution in S^5, which is orthogonal to ξ:

$$\Delta_z = \left\{v \in T_z S^5 \mid \langle\xi, v\rangle = 0\right\}.$$

We observe that Δ is invariant by the complex structure of $\mathbb{C}^3$.

Definition 1. The *contact angle* β is the complementary angle between the contact distribution Δ and the tangent space *TS* of the surface.

Let (e_1, e_2) be a local frame of *TS*, where $e_1 \in TS \cap \Delta$, where e_1 is the characteristic field and introduced by Bennequin in [1]. Then $\cos\beta = \langle\xi, e_2\rangle$. Finally, let v be the unit vector in the direction of the orthogonal projection of e_2 on Δ, defined by the following relation

$$e_2 = \sin\beta v + \cos\beta\xi. \tag{4}$$

Assume that the fundamental second form in the direction of a normal vector field e_3 is diagonal. Then we have the following connection forms:

$$\begin{aligned}
&\theta_1^3 = a\theta^1; \qquad \theta_2^3 = -a\theta^2,\\
&\theta_1^4 = \mathrm{d}\alpha + (-\sin\alpha\cot\beta)\theta^1 - a\csc\beta\theta^2,\\
&\theta_2^4 = \mathrm{d}\alpha\circ J - a\csc\beta\theta^1 - (-\sin\alpha\cot\beta)\theta^2,\\
&\theta_1^5 = -\cos\alpha\theta^2; \qquad \theta_2^5 = -\cos\alpha\theta^1.
\end{aligned} \tag{5}$$

Normal connection forms are:

$$\begin{aligned}
&\theta_3^4 = -\cot\alpha\csc\beta\,\mathrm{d}\alpha\circ J + a\cot\alpha\cot^2\beta\theta^1 + (-\cos\alpha\cot\beta\csc\beta + 2\sec\beta\cos\alpha)\theta^2,\\
&\theta_3^5 = (-\csc\beta\sin\alpha)\theta^1 - a\cot\beta\theta^2,\\
&\theta_4^5 = \cot\beta(\mathrm{d}\alpha\circ J) - a\cot\beta\csc\beta\theta^1 + \big(\sin\alpha\big(\cot^2\beta - 1\big)\big)\theta^2.
\end{aligned} \tag{6}$$

Using the connection forms (5) and (6) in the Codazzi–Ricci equations, we have

$$\mathrm{d}\theta_1^3 + \theta_2^3\wedge\theta_1^2 + \theta_4^3\wedge\theta_1^4 + \theta_5^3\wedge\theta_1^5 = 0.$$

This implies that

$$\begin{aligned}
&-a_2 + a^2\big(\cot\alpha\csc\beta\cot^2\beta\big) - a\cot\alpha\big(\csc^2\beta + \cot^2\beta\big)\alpha_2\\
&\quad - \cos\alpha\csc\beta\big(2(\cot\beta - \tan\beta)\alpha_1 - \sin\alpha\big(\cot^2\beta - 3\big)\big) + \cot\alpha\csc\beta|\nabla\alpha|^2 = 0.
\end{aligned} \tag{7}$$

Replacing the following (5) and (6) in the Codazzi–Ricci equations

$$\begin{aligned}
&\mathrm{d}\theta_2^3 + \theta_1^3\wedge\theta_2^1 + \theta_4^3\wedge\theta_2^4 + \theta_5^3\wedge\theta_2^5 = 0,\\
&\mathrm{d}\theta_1^4 + \theta_2^4\wedge\theta_1^2 + \theta_3^4\wedge\theta_1^3 + \theta_5^4\wedge\theta_1^5 = 0,\\
&\mathrm{d}\theta_3^5 + \theta_1^5\wedge\theta_3^1 + \theta_2^5\wedge\theta_3^2 + \theta_4^5\wedge\theta_3^4 = 0.
\end{aligned}$$

We get

$$a_1 + a(\cot\alpha\alpha_1 + 6\tan\beta\cos\alpha) - 2\sec\beta\cos\alpha\alpha_2 = 0. \tag{8}$$

Using the connection forms (5) and (6) in the Codazzi–Ricci equations

$$\begin{aligned}
&\mathrm{d}\theta_2^4 + \theta_1^4\wedge\theta_2^1 + \theta_3^4\wedge\theta_2^3 + \theta_5^4\wedge\theta_2^5 = 0,\\
&\mathrm{d}\theta_4^5 + \theta_1^5\wedge\theta_4^1 + \theta_2^5\wedge\theta_4^2 + \theta_3^5\wedge\theta_4^3 = 0,\\
&\mathrm{d}\theta_3^4 + \theta_1^4\wedge\theta_3^1 + \theta_2^4\wedge\theta_3^2 + \theta_5^4\wedge\theta_3^5 = 0.
\end{aligned}$$

We have

$$\begin{aligned}
&a_2 - a^2\big(\cot\alpha\sin\beta\cot^2\beta\big) + a\cot\alpha\alpha_2 + 2\cos\alpha(\cot\beta - 3\tan\beta)\\
&\quad + 2\cos\alpha\sin\beta(\cot\beta - \tan\beta)\alpha_1 + \sin\alpha\cos\alpha\sin\beta\big(5 - \cot^2\beta\big) + \sin\beta\Delta\alpha = 0.
\end{aligned} \tag{9}$$

The Codazzi–Ricci equations

$$\begin{aligned}
&\mathrm{d}\theta_1^2 + \theta_3^2\wedge\theta_1^3 + \theta_4^2\wedge\theta_1^4 + \theta_5^2\wedge\theta_1^5 = \theta^2\wedge\theta^1,\\
&\mathrm{d}\theta_1^5 + \theta_2^5\wedge\theta_1^2 + \theta_3^5\wedge\theta_1^3 + \theta_4^5\wedge\theta_1^4 = 0
\end{aligned}$$

give the following equation

$$\begin{aligned}
&a^2\big(1 + \csc^2\beta\big) - 2a\csc\beta\alpha_2 + |\nabla\alpha|^2 + 2\sin\alpha(\tan\beta - \cot\beta)\alpha_1 - 4\tan^2\beta\cos^2\alpha\\
&\quad - \sin^2\alpha\big(1 - \cot^2\beta\big) = 0.
\end{aligned} \tag{10}$$

The following Codazzi equation is automatically verified

$$\mathrm{d}\theta_2^5 + \theta_1^5\wedge\theta_2^1 + \theta_3^5\wedge\theta_2^3 + \theta_4^5\wedge\theta_2^4 = 0.$$

3. Proof of the results

Using Eqs. (7) and (9) we have:

$$\begin{aligned}\Delta(\alpha) &= \cot\alpha\csc^3(\beta)|\nabla\alpha|^2 + a^2\cot\alpha\cot^4(\beta) - 2a\cot\alpha\csc\beta\cot^2\beta\alpha_2\\ &\quad - 2\cos\alpha(\cot\beta-\tan\beta)\tan^2\beta\alpha_1 + \sin\alpha\cos\alpha\big(5-\cot^4\beta - 3\csc^2\beta\big).\end{aligned} \tag{11}$$

Using Eq. (10) we get a as the square root sign should cover all of $f(\alpha,\beta)$

$$a = \frac{2\csc\beta\alpha_2 + \sqrt{f}(\alpha,\beta)}{2(1+\csc^2\beta)}$$

where $f(\alpha,\beta)$ is given by

$$\begin{aligned}f(\alpha,\beta) &= 4\alpha_2^2\cot^2\beta - 4\big(1+\csc^2\beta\big)\alpha_1^2\\ &\quad - 4\big(1+\csc^2\beta\big)\big(2\sin\alpha(\tan\beta-\cot\beta)\alpha_1 - 4\tan^2\beta\cos^2\alpha - \sin^2\alpha\big(1-\cot^2\beta\big)\big).\end{aligned}$$

Namely, the frame field $e=(e_0,e_1,\dots,e_5): S\to \mathrm{SO}(6)$, where $e_0 = x$ gives rise to the differential 1-forms θ_j^i, $0\leqslant i,j\leqslant 5$, defined by the entries of the $o(6)$ form $\theta = e^{-1}\,\mathrm{d}e$. From the structure equations, $\mathrm{d}\theta = -\theta\wedge\theta$, which I call the Codazzi–Ricci equations, I derive the PDE (3), with Eq. (10) giving a as a function of α and β. Then, θ satisfies the structure equations of SO(6), and therefore there exist a map $e: S\to \mathrm{SO}(6)$ such that $e^{-1}\,\mathrm{d}e = \theta$ (assuming that S is simply connected). The desired immersion is then given by $x = e_0$, which prove Theorem 1.

3.1. Proof of Corollary 1

Suppose that $K=0$ in Eq. (2), we determine α_1 as a function of α, and from Eqs. (8) and (7), we get α as a function of β, and therefore constant, which prove Corollary 1.

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