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Lie Algebras/Partial Differential Equations

Sharp weighted Hardy type inequalities and Hardy–Sobolev type inequalities on polarizable Carnot groups ☆

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Abstract

In this Note, we establish sharp weighted Hardy type inequalities with a more general index p on polarizable Carnot groups, which include Kombe's recent results; then a weighted Hardy–Sobolev type inequality is obtained by using previous inequalities. ***To cite this article: J. Wang, P. Niu, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Inégalités optimales de type de Hardy avec poids et inégalités de type Hardy–Sobolev sur les groupes de Carnot polarisables. Dans cette Note nous établissons des inégalités optimales de type de Hardy avec poids dans le cas d'indices p plus généraux sur des groupes de Carnot polarisables ; nos résultats contiennent ceux obtenus récemment par Kombe. À partir des résultats précédemment établis nous démontrons une inégalité de type Hardy–Sobolev avec poids. ***Pour citer cet article : J. Wang, P. Niu, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction

It is well known that Hardy inequalities on the Euclidean space play an important role in the study of linear and nonlinear partial differential equations. In particular, sharp inequalities have attracted considerable attention because of their application to certain singular problems. There exists some literature dealing with Hardy type inequalities on Carnot groups.

Recently, Kombe in [4] obtained the following Hardy type inequalities on polarizable Carnot groups:

$$\int_G |\nabla_G \phi|^p \, \mathrm{d}x \geqslant \left(\frac{Q-p}{p}\right)^p \int_G \frac{|\nabla_G N|^p}{N^p} |\phi|^p \, \mathrm{d}x, \tag{1}$$

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where $\phi \in C_0^\infty(G\backslash\{0\})$, $Q \geqslant 3$, $1 < p < Q$ and N is the homogeneous norm associated with Folland's fundamental solution of sub-Laplacian (see [2]). Later in [5] the same author established the sharp weighted Hardy type inequalities for $p = 2$ on the same groups:

$$\int_G |\nabla_G \phi|^2 N^\alpha |\nabla_G N|^\gamma \, \mathrm{d}x \geqslant \left(\frac{Q+\alpha-2}{2}\right)^2 \int_G |\phi|^2 N^{\alpha-2} |\nabla_G N|^{2+\gamma} \, \mathrm{d}x, \tag{2}$$

where $\phi \in C_0^\infty(G\backslash\{0\})$, $\alpha \in \mathbb{R}$, $Q \geqslant 3$, $\gamma > -1$, and N is as before.

All of these works motivate us investigating the interesting question: Whether we can establish some new weighted inequalities containing (1) and (2).

Therefore, the first aim of this Note is to seek such inequalities. The idea of proof is inspired by Tyson [6] who proved the sharp weighted Young's inequality on Carnot groups.

The second aim of this paper is to deduce the weighted Hardy–Sobolev type inequality on polarizable Carnot groups. Han Y. and Niu P. in [3] showed a Hardy–Sobolev type inequality on groups of Heisenberg type by the method of function representations. Here, we directly gain the weighted Hardy–Sobolev inequality by using Sobolev inequalities in [2] and weighted Hardy type inequalities that we proved.

The structure of this Note is as follows: in Section 2, some basic facts on polarizable Carnot groups are collected. Section 3 is devoted to establishing the sharp weighted Hardy type inequality on polarizable Carnot groups. Finally, we test the weighted Hardy–Sobolev type inequality in Section 4.

2. Basic facts

We begin by gathering some preliminary facts on polarizable Carnot groups and refer interested readers to [1] for more precise information on this subject.

A Carnot group is a connected, simply connected, nilpotent Lie group $G = (\mathbb{R}^n, \circ)$ whose Lie algebra $\mathcal{G}$ permits a stratification. A homogeneous norm on G is a function $N : \mathbb{R}^n \to \mathbb{R}$, which is continuous in $\mathbb{R}^n$ and C^∞ outside the origin.

The sub-Laplacian on G is given by $\Delta_G = \nabla_G \cdot \nabla_G = \sum_{j=1}^n X_j^2$, where $\nabla_G = (X_1, \ldots, X_n)$ and $X_1, \ldots, X_n$ are the generators of $\mathcal{G}$. The fundamental solution u for Δ_G is defined to be a weak solution to the equation $-\Delta_G u = \delta$, where δ denotes the Dirac distribution with singularity at the neutral element 0 of G. In [2] Folland proved that in any Carnot group, there exists a homogeneous norm N such that $u = N^{2-Q}$ is a fundamental solution for Δ_G, where Q is the homogeneous dimension of G.

Balogh Z.M. and Tyson J.T. [1] introduced the concept of the polarizable Carnot group and constructed the polar coordinates on the group. Following [1], a Carnot group G is polarizable if the homogeneous norm $N = u^{\frac{1}{2-Q}}$ is ∞-harmonic in $G\backslash\{0\}$, that is, $\Delta_{G,\infty} N := \frac{1}{2}\langle \nabla_G(|\nabla_G N|^2), \nabla_G N\rangle = 0$ in $G\backslash\{0\}$. They pointed out that examples of polarizable Carnot groups include the usual Euclidean space, the Heisenberg group as well as groups of the Heisenberg type, and proved that the fundamental solution of the p-sub-Laplacian $\Delta_{G,p} f = \mathrm{div}_G(|\nabla_G f|^{p-2} \nabla_G f)$ with $f \in C^2(G)$ is given by $u_p = N^{\frac{p-Q}{p-1}}$ with $p \neq Q$ and $u_p = -\log N$ with $p = Q$.

3. Sharp weighted Hardy type inequalities

One of the main results is the following:

Theorem 3.1. *Let G be a polarizable Carnot group, $\alpha \in \mathbb{R}$, $\gamma > -1$, $1 < p < Q + \alpha$, $\phi \in C_0^\infty(G\backslash\{0\})$, and $N = u^{\frac{1}{2-Q}}$, then it holds:*

$$\int_G |\nabla_G \phi|^p N^\alpha |\nabla_G N|^\gamma \, \mathrm{d}x \geqslant \left(\frac{Q+\alpha-p}{p}\right)^p \int_G |\phi|^p N^{\alpha-p} |\nabla_G N|^{p+\gamma} \, \mathrm{d}x. \tag{3}$$

Moreover $(\frac{Q+\alpha-p}{p})^p$ is the best constant.

In order to prove (3), we need the following result first:

Lemma 3.2. *For $p \neq Q+\alpha$, $u_p = N^{\frac{p-Q-\alpha}{p-1}}$ is a weighted p-harmonic function, that is, u_p satisfies,*

$$\mathrm{div}_G\big(|\nabla_G u_p|^{p-2} N^\alpha |\nabla_G N|^\gamma \nabla_G u_p\big) = \sum_{i=1}^n X_i\big(|\nabla_G u_p|^{p-2} N^\alpha |\nabla_G N|^\gamma X_i u_p\big) = 0. \tag{4}$$

Proof. We write $A \sim B$ if $A = c \cdot B$ for some $c = c(Q, p, \alpha)$. Noting $\nabla_G u_p = c \cdot N^{\frac{1-Q-\alpha}{p-1}} \nabla_G N$, that is, $\nabla_G u_p \sim N^{\frac{1-Q-\alpha}{p-1}} \nabla_G N$. We have:

$$\mathrm{div}_G\big(|\nabla_G u_p|^{p-2} N^\alpha |\nabla_G N|^\gamma \nabla_G u_p\big) \sim \mathrm{div}_G\big(|\nabla_G N|^{p+\gamma-2} N^{1-Q} \nabla_G N\big). \tag{5}$$

Then, using $\mathrm{div}_G(\frac{N}{|\nabla_G N|^2} \nabla_G N) = Q$ and $\Delta_{0,\infty} N = 0$ on polarizable Carnot groups leads to

$$\mathrm{div}_G\big(|\nabla_G N|^{p+\gamma-2} N^{1-Q} \nabla_G N\big) = Q\frac{|\nabla_G N|^{p+\gamma}}{N^Q} + (p+\gamma)\frac{|\nabla_G N|^{p+\gamma-4}}{N^{Q-1}} \Delta_{0,\infty} N - Q\frac{|\nabla_G N|^{p+\gamma}}{N^Q} = 0.$$

This ends the proof. □

Proof of Theorem 3.1. From Lemma 3.2, we know that $N^{\frac{p-Q-\alpha}{p-1}}$ is a (weak) solution of (4), i.e.

$$\int_G |\nabla_G u_p|^{p-2} \langle \nabla_G u_p, \nabla_G \varphi\rangle N^\alpha |\nabla_G N|^\gamma = 0, \tag{6}$$

where $\varphi \in C_0^\infty(G)$. Now let us take $\varphi = |\phi|^p N^{Q+\alpha-p}$. Then

$$\int_G \big|\nabla_G\big(N^{\frac{Q+\alpha-p}{p-1}}\big)\big|^{p-2} \nabla_G\big(N^{\frac{Q+\alpha-p}{p-1}}\big) N^\alpha |\nabla_G N|^\gamma \cdot \nabla_G\big(|\phi|^p N^{Q+\alpha-p}\big) = 0, \tag{7}$$

i.e.

$$(Q+\alpha-p)\int_G N^{\alpha-p} |\nabla_G N|^{p+\gamma} |\phi|^p + p\int_G N^{1+\alpha-p} |\nabla_G N|^{p+\gamma-2} |\phi|^{p-2} \phi \nabla_G N \cdot \nabla_G \phi = 0. \tag{8}$$

By Hölder's inequality we have

$$\frac{Q+\alpha-p}{p}\int_G N^{\alpha-p} |\nabla_G N|^{p+\gamma} |\phi|^p \leqslant \bigg[\int_G N^{\alpha-p} |\nabla_G N|^{p+\gamma} |\phi|^p\bigg]^{\frac{p-1}{p}} \bigg[\int_G N^\alpha |\nabla_G N|^\gamma |\nabla_G \phi|^p\bigg]^{\frac{1}{p}},$$

and then (3) is obtained.

Next we show that the constant $(\frac{Q+\alpha-p}{p})^p$ in (3) is sharp. Letting $\varepsilon > 0$, we consider the radial function $u_\varepsilon = C_\varepsilon$ with $0 \leqslant N \leqslant 1$ and $u_\varepsilon = C_\varepsilon N^{\frac{p-Q-\alpha}{p}-\varepsilon}$ with $N > 1$, where $C_\varepsilon = (\frac{Q+\alpha-p}{p} + \varepsilon)^{-1}$. It is easy to see that $\nabla_G u_\varepsilon = 0$ with $0 \leqslant N \leqslant 1$ and $\nabla_G u_\varepsilon = -N^{-\frac{Q+\alpha+p\varepsilon}{p}} \nabla_G N$ with $N > 1$. So

$$\begin{aligned}\int_G N^{\alpha-p} |\nabla_G N|^{p+\gamma} |u_\varepsilon|^p &= \int_{B_1} N^{\alpha-p} |\nabla_G N|^{p+\gamma} |u_\varepsilon|^p + \int_{G\setminus B_1} N^{\alpha-p} |\nabla_G N|^{p+\gamma} |u_\varepsilon|^p \\ &= C_\varepsilon^p\bigg(\int_{B_1} N^{\alpha-p} |\nabla_G N|^{p+\gamma} + \int_{G\setminus B_1} N^\alpha |\nabla_G N|^\gamma |\nabla_G u_\varepsilon|^p\bigg) \\ &= C_\varepsilon^p\bigg(\int_{B_1} N^{\alpha-p} |\nabla_G N|^{p+\gamma} + \int_G N^\alpha |\nabla_G N|^\gamma |\nabla_G u_\varepsilon|^p\bigg).\end{aligned}$$

Using the polar coordinate formula for polarizable Carnot groups [1] and letting $\varepsilon \to 0$, the conclusion holds. □

Remark 3.3. Let $\alpha = \gamma = 0$ in (3), then we immediately have (1). Letting $p = 2$ in (3), then (2) holds.

4. Weighted Hardy–Sobolev type inequalities

This section is devoted to establishing the Hardy–Sobolev inequality on polarizable Carnot groups. Our method is different from that by Han Y. and Niu P. [3].

Theorem 4.1. *Let G be a polarizable Carnot group, $\alpha \in R$, $\gamma > -1$, $1 < p < Q + \alpha$, $\phi \in C_0^\infty(G\backslash\{0\})$, $N = u^{\frac{1}{2-Q}}$, $Q \geqslant 3$ and $0 \leqslant s \leqslant p$. Then*

$$\int_G \left(\frac{|\nabla_G N|^{1+\frac{\gamma}{p}}}{N^{1-\frac{\alpha}{p}}}\right)^s |\phi|^{\frac{p(Q-s)}{Q-p}} \leqslant \left(\frac{p}{Q+\alpha-p}\right)^s s_q^{-\frac{Q(p-s)}{Q-p}} \left[\int_G |\nabla_G \phi|^p N^\alpha |\nabla_G N|^\gamma\right]^{\frac{s}{p}} \left[\int_G |\nabla_G \phi|^p\right]^{\frac{Q(p-s)}{p(Q-p)}}, \tag{9}$$

where s_q is the constant in the Sobolev inequality (see (10) below).

We recall the Sobolev inequality:

Lemma 4.2. *Let G be a Carnot group, $\phi \in C_0^\infty(G\backslash\{0\})$ and $1 < p < Q$. Then the following inequality holds:*

$$\left(\int_G |\nabla_G \phi|^p\right)^{\frac{1}{p}} \geqslant s_q \left(\int_G |\phi|^{\frac{pQ}{Q-p}}\right)^{\frac{Q-p}{pQ}}. \tag{10}$$

Proof of Theorem 4.1. It is easy to show

$$\int_G \left(\frac{|\nabla_G N|^{1+\frac{\gamma}{p}}}{N^{1-\frac{\alpha}{p}}}\right)^s |\phi|^{\frac{p(Q-s)}{Q-p}} \leqslant \left[\int_G \frac{|\nabla_G N|^{p+\gamma}}{N^{p-\alpha}} |\phi|^p\right]^{\frac{s}{p}} \left[\int_G |\phi|^{\frac{pQ}{Q-p}}\right]^{\frac{p-s}{p}}. \tag{11}$$

By (3),

$$\left[\int_G \frac{|\nabla_G N|^{p+\gamma}}{N^{p-\alpha}} |\phi|^p\right]^{\frac{s}{p}} \leqslant \left(\frac{p}{Q+\alpha-p}\right)^s \left[\int_G |\nabla_G \phi|^p N^\alpha |\nabla_G N|^\gamma\right]^{\frac{s}{p}}. \tag{12}$$

Using (10),

$$\left[\int_G |\phi|^{\frac{pQ}{Q-p}}\right]^{\frac{p-s}{p}} \leqslant s_q^{-\frac{(p-s)Q}{Q-p}} \left[\int_G |\nabla_G \phi|^p\right]^{\frac{(p-s)Q}{(Q-p)p}}. \tag{13}$$

Combining (11), (12) and (13), Eq. (9) follows. □

Remark 4.3. If $s = 0$ in (9), then one has (10). In the case $s = p$, (9) becomes (3).

Remark 4.4. (9) is valid for groups of Heisenberg type too. It means our inequalities involve Han and Niu's results [3] essentially.

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