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Numerical Analysis

Adaptive mesh for algebraic orthogonal subscale stabilization of convective dispersive transport

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Abstract

We derive a residual a posteriori error estimator for the algebraic orthogonal subscales stabilization of convective dispersive transport equation. The estimator yields upper bound on the error which is global and lower bound that is local. Numerical studies show the behaviour of the error indicator and how it is robust to deal with singularities. ***To cite this article: B. Achchab et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Adaptation de maillage pour l'équation de convection dispersion stabilisée par la méthode algébrique de sous-mailles orthogonales. On développe un estimateur d'erreur a posteriori pour l'équation de convection dispersion stabilisée par la méthode algébrique de sous-mailles orthogonales. On obtient une majoration et une minoration de l'erreur. Les résultats numériques montre l'efficacité de l'indicateur d'erreur dans les régions des singularités où la solution présente des couches limites. ***Pour citer cet article : B. Achchab et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction and model description

Miscible displacement of a solute in soil solutions is simulated by convection dispersion equation (CDE). The main difficulties when simulating numerically this process is the convective-dominated transport in porous media. Classical numerical methods either lack stability, resulting in non-physical instabilities, or accuracy, when excessive numerical diffusion is produced. Stabilized finite element methods [3,4,7–9,11] are particularly interesting for those cases. In this work we employ the algebraic orthogonal subscales (AOSS) method plus adaptive mesh refinement to solve in a domain $\Omega \in \mathbf{R}^2$, with a Lipschitz-continuous boundary $\partial\Omega$, the following concentration equation:

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$$\nabla \cdot (\nu c - D\nabla c) = \gamma - \mu c \quad \text{in } \Omega, \tag{1}$$

$$c = 0 \quad \text{on } \partial\Omega = \Gamma, \tag{2}$$

where $c = c(x, y)$ is the concentration at location (x, y); $D = \alpha_L|\nu|$ is the dispersion coefficient; α_L is the longitudinal dispersivity; $\nu = q/\theta$ is the pore water velocity; θ is the volumetric water content; q is flux density of soil water (or the Darcy velocity) and γ is the production rate. We suppose that the pore water velocity ν is solenoidal.

We define the operator $\mathcal{L}c = \nabla \cdot (\nu c - D\nabla c) + \mu c$. The weak form of the problem is to seek $c \in H_0^1(\Omega)$, such that

$$a(c, w) = l(w) \quad \forall w \in H_0^1(\Omega), \tag{3}$$

where $a(c, w) = D(\nabla c, \nabla w) + (\nu \cdot \nabla c + \mu c, w)$ and $l(w) = (\gamma, w)$.

2. Algebraic orthogonal subscales stabilization

Let $\mathcal{V}_{h,0} \subset H_0^1(\Omega)$, be a conforming finite element space of piecewise polynomials. The standard Galerkin approximation of (3) is to find $c_h \in \mathcal{V}_h$, such that

$$a(c_h, w_h) = l(w_h) \quad \forall w_h \in \mathcal{V}_{h,0}. \tag{4}$$

The standard Galerkin method lacks stability for near-hyperbolic problem, as shown in Fig. 1.

The key idea of the multiscale formulation [9] is to consider $\mathcal{V}_0 = H_0^1(\Omega)$ as the direct sum of two spaces

$$\mathcal{V}_0 = \mathcal{V}_{h,0} \oplus \widetilde{\mathcal{V}}, \tag{5}$$

where $\mathcal{V}_{h,0}$ is the space of resolved scales and $\widetilde{\mathcal{V}}$ is the space of subgrid scales. We can now split the problem (4):

$$a(c_h + \tilde{c}, w_h) = l(w_h) \quad \forall w_h \in \mathcal{V}_{h,0}, \tag{6}$$

$$a(c_h + \tilde{c}, \tilde{w}) = l(\tilde{w}) \quad \forall \tilde{w} \in \widetilde{\mathcal{V}}. \tag{7}$$

The subscales are modeled analytically using an algebraic orthogonal subscales (AOSS) approximation [7], $\tilde{c} = \tau \mathcal{R} c_h$, where $\tau = (4\frac{D}{h^2} + 2\frac{|\nu|}{h} + \mu)^{-1}$ is called the relaxation time and $\mathcal{R}c_h := \gamma - \mathcal{L}c_h$ is the grid scale residual. After integration by parts on each element, the equation for the grid scales reads:

$$a(c_h, w_h) + \sum_{T \in \mathcal{T}_h} \left(\tilde{c}, \mathcal{L}^* w_h\right) = l(w_h) \quad \forall w_h \in \mathcal{V}_{h,0}, \tag{8}$$

where $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$. The final equation for the resolved scales includes the usual Galerkin terms and some additional volume integrals evaluated element by element. Since the subscales $\tilde{c}$ are proportional to the grid scale residual, the method is residual-based and therefore, automatically consistent.

3. Adaptive strategy

For a subset $S \subset \Omega$, let $\|\cdot\|_{0,S}$ to be the usual L^2 norm. The purpose of this Note is to get local lower and global upper bounds for the error measured in the energy norm:

$$\|v\|_S^2 = D\|\nabla v\|_{0,S}^2 + \|v\|_{0,S}^2.$$

Let $\mathcal{T}_h$ be a triangulation of Ω and $\mathcal{E}_h$ denote the set of all $(n-1)$-faces in $\mathcal{T}_h$. This set can be split into $\mathcal{E}_h = \mathcal{E}_{h,\Omega} \cup \mathcal{E}_{h,D}$, where $\mathcal{E}_{h,\Omega}$ and $\mathcal{E}_{h,D}$ refer to interior faces and Dirichlet boundary faces, respectively. For all $E \in \mathcal{E}_{h,\Omega}$ and for all ϕ piecewise smooth, $[\phi]_E$ denotes the jump of ϕ across E (the sign of this quantity is irrelevant in the sequel). For all $S \in \mathcal{T}_h \cup \mathcal{E}_h$, let $\alpha_S = \min\{h_S D^{-\frac{1}{2}}, 1\}$, where h_S denotes the diameter of S. Denote by γ_h, ν_h, and μ_h the L^2-projection of the data γ, ν, and μ onto the space of piecewise constant functions on $\mathcal{T}_h$. Define the elementwise residual estimators as

$$\eta_T^2 = \alpha_T^2 \|r_T\|_{0,T}^2 + \sum_{E \in \mathcal{E}_h;\ E \subset \partial T} D^{-\frac{1}{2}} \alpha_E \|r_E\|_{0,E}^2, \tag{9}$$

where

$$r_T = \gamma_h + D\triangle c_h - \nu_h \cdot \nabla c_h - \mu_h c_h, \quad \text{and} \quad r_E = \begin{cases} [D\partial_n c_h]_E & \text{if } E \in \mathcal{E}_{h,\Omega}, \\ 0 & \text{if } E \in \mathcal{E}_{h,D}. \end{cases} \tag{10}$$

Finally, define the elementwise data oscillation estimator as

$$\Theta_T^2 = \alpha_T^2 \left\| (\gamma - \gamma_h) + (\nu - \nu_h) \cdot \nabla c_h + (\mu - \mu_h) c_h \right\|_{0,T}^2.$$

For $T \in \mathcal{T}_h$, let $\omega_T = \bigcup\{T' \in \mathcal{T}_h | T \cap T' \neq \emptyset\}$ and set $\Theta_{\omega_T}^2 = \sum_{T' \in \omega_T} \Theta_{T'}^2$.

We proceed with the same strategy as in our work for the subgrid viscosity stabilization [1] using the estimates of the Clément operator [6] and the techniques of bubble functions [12] we obtain the following results:

Theorem 1. *Let c and c_h be the unique solutions of* (3) *and* (8) *respectively. Then*

$$\|c - c_h\|_\Omega^2 \preceq \sum_{T \in \mathcal{T}_h} \left[\eta_T^2 + \Theta_T^2\right], \tag{11}$$

and for all $T \in \mathcal{T}_h$,

$$\eta_T \preceq \left(1 + \varepsilon^{-1/2} \alpha_T\right) \|c - c_h\|_{\omega_T} + \Theta_{\omega_T}. \tag{12}$$

Algorithm *(Maximum strategy). For the mesh refinement we use criterion of the maximum strategy.*

I. *Given an initial mesh $\mathcal{T}_h$ that is quasi-uniform* [5]*, we compute the indicator η_T for each $T \in \mathcal{T}_h$.*
II. *Put* $\eta = \max_{T \in \mathcal{T}_h}$.
III. *Then a subset $\mathcal{S}_h$ of marked elements should be refined if $\eta_T \geqslant \delta\eta$ for each $T \in \mathcal{S}_h$, where δ is a threshold* $(0 < \delta < 1)$*. Here we choose* $\delta = 0.25$.

4. Numerical studies and concluding remarks

Under simplifying assumptions, we solve the problem (using FreeFem software [2] and Xd3d [10]) with parameters $\nu = (1, 1)^T$, $D = 10^{-4}$, $\mu = 1$ and we take γ such that the exact solution is:

$$c(x, y) = xy(1 - x)(1 - y)\left[1 + \tanh\left(100\left(x^2 + y^2 - 0.25\right)\right)\right].$$

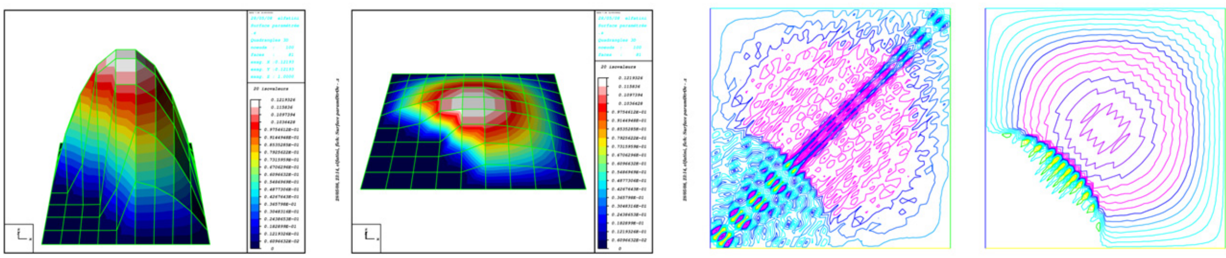

Fig. 1. From left to right: two views of exact solution, its standard Galerkin approximation and the corresponding AOSS stabilization.

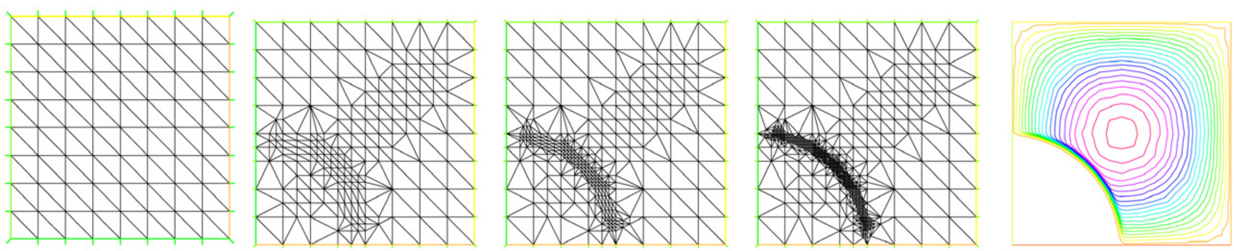

Fig. 2. Successive mesh refinements and corresponding solution.

In Fig. 1 it is observed that standard Galerkin finite element produces a globally oscillatory solution, while the solution obtained by the (AOSS) method is non-oscillatory except in the vicinity of interior layers. Owing to the local error indicator η_T and using the above adaptive algorithm criterion; the estimators capture the remaining oscillations. In Fig. 2, adaptive meshes and the corresponding solution are presented.

Especially the estimators can be used to construct adaptive meshes and then to deal with singularities in the boundary or interior layers.

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References

[1] B. Achchab, M. El Fatini, A. Ern, A. Souissi, A posteriori error estimates for subgrid viscosity stabilized approximations of convection–diffusion equations, Appl. Math. Lett. (2008), submitted for publication.
[2] D. Bernardi, F. Hecht, K. Ohtsuka, O. Pironneau, FreeFem+ for Macs, Pcs, Linux (Dec 25 2008) http://www.freefem.org.
[3] F. Brezzi, A. Russo, Choosing bubbles for advection–diffusion problems, Math. Models Methods Appl. Sci. 4 (1994) 571–587.
[4] A.N. Brooks, T.J.R. Hughes, Streamline upwind/Petrov Galerkin formulation for convection dominated flows with particular emphasis on the incompressible Navier–Stokes equations, Model. Comput. Methods Appl. Mech. Engrg. 32 (1-3) (1982) 199.
[5] P.G. Ciarlet, The Finite Element Methods for Elliptic Problems, North-Holland, Amsterdam, 1978.
[6] P. Clément, Approximation by finite element functions using local regularization, RAIRO, Anal. Numér. 9 (1975) 77–84.
[7] R. Codina, On stabilized finite element methods for linear systems of convection–diffusion–reaction equations, Comp. Meth. Appl. Mech. Engrg. 188 (2000) 61–82.
[8] J.L. Guermond, Subgrid Stabilization of Galerkin approximations of linear monotone operators, IMA J. Numer. Anal. 21 (2001) 165–197.
[9] T.J.R. Hughes, Multiscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation, subgrid-scale models, bubbles and the origin of stabilized methods, Comput. Meth. Appl. Mech. Engrg. l 27 (1995) 387–401.
[10] F. Jouve, Xd3d-Version 7.72, Visualisation de maillages 2D et 3D et de surfaces 3D sous X, http://www.cmap.polytechnique.fr.
[11] O. Pironneau, On the transport–diffusion algorithm and its applications to the Navier–Stokes equations, Numer. Math. 38 (1982) 309–332.
[12] R. Verfürth, A posteriori error estimators for convection–diffusion equations, Numer. Math 80 (1998) 641–663.