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Algebraic Geometry

A Note on Seshadri constants on general $K3$ surfaces ☆

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Abstract

We prove a lower bound on the Seshadri constant $\varepsilon(L)$ on a $K3$ surface S with $\operatorname{Pic} S \simeq \mathbb{Z}[L]$. In particular, we obtain that $\varepsilon(L) = \alpha$ if $L^2 = \alpha^2$ for an integer α. ***To cite this article: A.L. Knutsen, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Une Note sur les constantes de Seshadri sur surfaces $K3$ générales. Nous démontrons une borne inférieure sur la constante de Seshadri $\varepsilon(L)$ sur un surface $K3$ telle que $\operatorname{Pic} S \simeq \mathbb{Z}[L]$. En particulier, nous obténons que $\varepsilon(L) = \alpha$ si $L^2 = \alpha^2$ pour un nombre entier α. ***Pour citer cet article : A.L. Knutsen, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Introduction and results

Let X be a smooth projective variety and L be an ample line bundle on X. Then the real number

$$\varepsilon(L,x) := \inf_{C \ni x} \frac{L.C}{\operatorname{mult}_x C},$$

introduced by Demailly [6], is the *Seshadri constant of L at $x \in X$* (where the infimum is taken over all irreducible curves on X passing through x). The (*global*) *Seshadri constant of L* is defined as

$$\varepsilon(L) := \inf_{x \in X} \varepsilon(L,x).$$

We refer to [8, pp. 270–303] for more background, properties and results on these constants.

The subtle point about Seshadri constants is that their exact values are known only in a few cases and even on surfaces it is difficult to control them.

It is known that the global Seshadri constant on a surface satisfies $\varepsilon(L) \leqslant \sqrt{L^2}$, cf. e.g. [10, Rem. 1], and that $\varepsilon(L)$ is rational if $\varepsilon(L) < \sqrt{L^2}$, cf. [11, Lemma 3.1] or [9, Cor. 2]. (It is not known whether Seshadri constants are always rational, but no examples are known where they are irrational.)

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In the case of $K3$ surfaces, Seshadri constants have only been computed for the hyperplane bundle of quartic surfaces [2] and in the particular case of non-globally generated ample line bundles [3, Prop. 3.1].

In this Note we prove the following result:

Theorem. *Let S be a smooth, projective $K3$ surface with* $\operatorname{Pic} S \simeq \mathbb{Z}[L]$. *Then either*

$$\varepsilon(L) \geqslant \lfloor \sqrt{L^2} \rfloor,$$

or

$$\left(L^2, \varepsilon(L)\right) \in \left\{ \left(\alpha^2 + \alpha - 2, \alpha - \frac{2}{\alpha+1}\right), \left(\alpha^2 + \frac{1}{2}\alpha - \frac{1}{2}, \alpha - \frac{1}{2\alpha+1}\right) \right\} \tag{1}$$

for some $\alpha \in \mathbb{N}$. (*Note that in fact $\alpha = \lfloor \sqrt{L^2} \rfloor$.*)

It is well known that a general polarized $K3$ surface has Picard number one (cf. [7, Thm. 14]).

Remark. In the two exceptional cases (1) of the theorem, the proof below shows that there has to exist a point $x \in S$ and an irreducible rational curve $C \in |L|$ (resp. $C \in |2L|$) such that C has a singular point of multiplicity $\alpha + 1$ (resp. $2\alpha + 1$) at x and is smooth outside x, and $\varepsilon(L) = L.C/\operatorname{mult}_x C$.

By a well-known result of Chen [4], rational curves in the primitive class of a *general* $K3$ surface in the moduli space are nodal. *Hence the first exceptional case in* (1) *cannot occur on a general $K3$ surface in the moduli space* (as $\alpha \geqslant 2$). If $\alpha = 2$, so that $L^2 = 4$, this special case is case (b) in [2, Theorem].

As one also expects that rational curves in any multiple of the primitive class on a *general* $K3$ surface are always nodal (cf. [5, Conj. 1.2]), we expect that also the second exceptional case in (1) cannot occur on a *general* $K3$ surface.

Since $\varepsilon(L) \leqslant \sqrt{L^2}$, an immediate corollary of the theorem is the following:

Corollary. *Let S be a smooth, projective $K3$ surface such that* $\operatorname{Pic} S \simeq \mathbb{Z}[L]$ *with $L^2 = \alpha^2$ for an integer $\alpha \geqslant 4$.*
Then $\varepsilon(L) = \alpha$.

2. Proof of the theorem

The reader will recognize the similarity of the proof of the theorem with the proofs of [1, Thm. 4.1] and [10, Prop. 1].

Set $\alpha := \lfloor \sqrt{L^2} \rfloor$ and assume that $\varepsilon(L) < \alpha$. Then it is well known (see e.g. [9, Cor. 2]) that there is an irreducible curve $C \subset S$ and a point $x \in C$ such that

$$C.L < \alpha \operatorname{mult}_x C. \tag{2}$$

Set $m := \operatorname{mult}_x C$. Since a point of multiplicity m causes the geometric genus of an irreducible curve to drop at least by $\binom{m}{2}$ with respect to the arithmetic genus, we must have

$$p_a(C) = \frac{1}{2}C^2 + 1 \geqslant \binom{m}{2} = \frac{1}{2}m(m-1), \tag{3}$$

so that

$$m(m-1) - 2 \leqslant C^2. \tag{4}$$

We have that $C \in |nL|$ for some $n \in \mathbb{N}$. From (2) we obtain $nL^2 < m\alpha$, so that, by assumption, $n\alpha^2 < m\alpha$, whence $n\alpha < m$. As $n\alpha \in \mathbb{Z}$ we must have

$$n\alpha \leqslant m - 1. \tag{5}$$

Combining (2), (4) and (5), we obtain

$$m(m-1) - 2 \leqslant C^2 = nC.L < n\alpha m \leqslant m(m-1),$$

giving the only possibilities $C^2 = n^2 L^2 = m(m-1) - 2$ and $n\alpha = m - 1$. It follows from (3) that C is a rational curve with a single singular point x of multiplicity $m \geqslant 2$.

As

$$C.L = nL^2 = \frac{m(m-1)-2}{n} = m\alpha - \frac{2}{n} \tag{6}$$

and $m\alpha \in \mathbb{Z}$, we must have $\frac{2}{n} \in \mathbb{Z}$, so that $n = 1$ or 2.

If $n = 1$, then $m = \alpha + 1$, so that $L^2 = C^2 = m(m-1) - 2 = \alpha(\alpha+1) - 2$ and $\varepsilon(L) = C.L/m = \alpha - \frac{2}{\alpha+1}$ from (6).

If $n = 2$, then $m = 2\alpha + 1$, so that $L^2 = \frac{1}{4}C^2 = \frac{1}{4}((2\alpha+1)2\alpha - 2)$ and $\varepsilon(L) = \alpha - \frac{1}{2\alpha+1}$ from (6).

This concludes the proof of the theorem.

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